Uncertain Graphs and Network Optimization

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Preface

There exist many network optimization problems in real life. Without historical data to estimate a probability distribution, decisions have to be made based on experts' belief degree. To better deal with such indeterminacy, an uncertainty theory was founded by Liu [44] in 2007 and then became a branch of mathematics for modeling uncertainty by belief degree. Within the framework of uncertainty theory, uncertain programming is a powerful tool to handle decision problems with uncertain variables.

Graph and Network

Graph is one of the most frequently used models in natural and artificial structures. For example, a city traffic graph can be used to describe the route conditions between urban roads. Relationship graphs can be used to describe the relationship between humans. What is network optimization? Network optimization is the study of how to design, management, control and optimize the network efficiently. Network optimization is a subset of mathematical optimization that is related to operations research, graph theory, and computational complexity theory. Many network optimization problems, such as the Chinese postman problem, optimal assignment problem, shortest path problem, maximum flow problem, can be studied on a graph or network. Chapter 1 introduces some basic concepts about graph and network.

Uncertainty Theory

We usually make a decision in the state of uncertainty. In order to deal with the uncertainty generated by human behavior, uncertainty theory was founded based on normality, duality, subadditivity, and product axioms. Up to now, uncertainty theory has been developed steadily and has become a branch of mathematics. Chapter 2 is devoted to uncertainty theory.

Uncertain Programming

As an important application of uncertainty theory, uncertain programming was founded in 2009. Since then, uncertain programming has been and continues to be an efficient tool for dealing with uncertain network optimization problems. Chapter 3 will present three types of uncertain programming to deal with optimization problems with uncertain variables: (1) the expected

value model; (2) chance-constrained programming; and (3) measure-chance programming.

Uncertain Graph

An uncertain graph is essentially a graph with uncertain edges. Thus an uncertain graph is usually used to model uncertain phenomena that vary with the topology structure. Chapter 4 is devoted to basic concepts of uncertain graphs as well as the connectivity index, Euler index, matching index and matching number.

Uncertain Network Optimization

Uncertain network optimization is the study of network optimization with uncertain variables which we often meet when making decisions are under uncertain conditions. Chapter 5 will introduce some classical network optimization problems within the framework of uncertainty theory, such as Chinese postman problem, optimal assignment problem and shortest path problem.

Applications of Uncertain Network Optimization

Chapter 6 will introduce the applications of uncertain network optimization in the field of transportation problems, dispatching medical supplies and the covering location problem.

Uncertainty Theory Online

If you want to read more papers related to uncertain graph and uncertain network, please visit the website at http://www.starfuture.net.cn:2007/online/.

Purpose

The purpose of this book is to provide a tool for the readers to deal with network optimization problems with uncertainty by belief degree. The book is suitable for researchers, engineers, and students in the field of mathematics, information science, operations research, industrial engineering, computer science, artificial intelligence, economics, and management science.

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We live in a world full of networks and uncertainties.

Chapter 0

Introduction

We usually make a decision in the state of indeterminacy. Without sufficient data, we have no choice but to obtain belief degrees that are evaluated by experts. For modeling such indeterminacy rationally, the uncertainty theory was founded by Liu [44]. Uncertain graphs refers to graphs with uncertain edges. Some interesting topics, such as the connectivity problem, Euler tour problem and matching problem, have been already studied using uncertainty theory. As an important application of uncertainty theory, uncertain programming was introduced by Liu [45] to deal with decision making problems involving uncertain variables. So far, uncertain programming has been widely applied in many optimization problems such as the Chinese postman problem, optimal assignment problem, shortest path problem, transportation problem, dispatching medical supplies, location problem and so on.

0.1 Graph and Network

In many real-world situations, the relationship between pairs of certain given objects can be described by means of a graph consisting of a set of vertices together with edges. For example, the objects could represent cities and the edges represent the highways joining pairs of cities; or the objects might be people, with edges joining pairs of friends.

It is quite well known that graph theory is the study of graphs. Since it originated in the seven bridges problem of Königsberg by Euler in 1736, graph theory has received great attention from many scholars, especially in the fields of mathematics and computer science. For example, the concept of a tree was implicitly proposed in the work of Kirchhoff. After that, many scholars employed the tree model to real problems such as enumerate chemical molecules. Other related topics such as connectivity problems, matching problems and regularity problems have also been studied by many researchers. The interested readers may refer to Chiue-Shieh [7], Costa-de

Werra-Picouleau [9], Edmonds-Johnson [16], He-Wei-Ye-Zhai [34], Jia-Yuan-Zhang [37], and Mynhardt-Teshima-Roux [54].

Generally speaking, a network is a connected weighted graph. In a network, each edge is assigned a real number. In different scenarios, the weights may have different meanings. In transportation networks, the weights may represent transportation times or running distances. In human resources networks, the weights may represent benefits. Network flow problem is one of the hottest problems in network optimization. For the classical works of network flow problem, the interested scholars can refer to Ford-Fulkerson [18] and Ahuja-Magnanti-Orlin [1]. For more research of graph theory and network optimization, the readers may consult Bondy-Murty [4] and Xu [68].

0.2 Uncertain Graph

An uncertain graph refers to a graph with uncertain factors, first proposed by Gao-Gao [22] in 2013. In a classic graph, the edges and vertices are all deterministic, i.e., either they exist or not. However, in an uncertain graph, some edges exist with some degrees in uncertain measure. Gao-Gao [22] first discussed the connectivity of uncertain graphs. After that, Gao-Qin [26] investigated the edge-connectivity of uncertain graphs. In practical applications, someone may be interested in whether or not two specific vertices are connected. Thus Zhang-Peng [73] introduced the concept of the connectedness strength of two vertices in an uncertain graph. They also discussed the relationship between the connectedness strength of two vertices and the connectedness index of the uncertain graph, which leads to a new method to get the connectivity index of an uncertain graph. In addition, Zhang-Peng [71] studied how likely an uncertain graph is Eulerian, and introduced an Euler index of an uncertain graph. In order to investigate the matching properties of uncertain graphs, a matching index was proposed by Zhang-Peng [74]. After that, Li-Zhang-Peng [41] investigated the uncertain measure that an uncertain graph is k-edge matching, which extended the work of Zhang-Peng [74] from a special case to general cases. Roysida-Peng-Chen-Widodo [58] discussed the chromatic number of an uncertain graph based on α-cut coloring. Furthermore, some other interesting topics such as cycle (Gao [20]), regularity (Gao [21]), diameter (Gao-Yang-Li-Kar [28]) of uncertain graphs were studied by related scholars.

0.3 Uncertain Network Optimization

Since uncertain programming was founded by Liu [45] in 2009, it has been widely employed in many uncertain network optimization problems. For instance, Gao [23] investigated the shortest path problem with uncertain arc lengths. Zhang-Peng [72] studied the Chinese postman problem on an uncertain network, introducing three types of optimal tour models for the

Chinese postman problem with uncertain edge weights. Zhang-Peng [75] investigated the optimal assignment problem in an uncertain environment. In addition, many other network optimization problems, such as the maximum flow problem (Han-Peng-Wang [32]), newsboy problem (Ding [11], Ding-Gao [12]), inventory problem (Qin-Kar [57], Sheng-Zhu-Wang [59], Wang-Qin-Kar [63]), contract design problem (Wu-Zhao-Tang [66], Liu-Zhao-Liu-Chen [51], Li-Liu-Chen [43]) and so on, have been studied in uncertain environments.

For uncertain transportation problems, Sheng-Yao [60] first introduced an uncertain transportation model with uncertain costs and demands. Then, as extensions of the transportation problem, Sheng-Yao [61] investigated a fixed charge transportation problem with uncertain factors, and Cui-Sheng [10] investigated a solid transportation in an uncertain environment. In addition, Zhang-Peng-Li-Chen [76] considered a fixed charge solid transportation problem under uncertainty, and introduced three novel models for the problem. Furthermore, Mou-Zhou-Chang [52] introduced a multi-objective model for a transportation problem with uncertain truck times and unit costs, and designed a solution algorithm for the proposed model based on a stepwise optimization strategy. Yang-Liu-Li-Gao-Ralescu [69] introduced type-2 uncertain optimization methods to the fixed charge solid transportation problem. Gao-Kar [25] further considered a solid transportation problem with product blending, in which the fixed charges and transportation costs were assumed to be uncertain variables.

For vehicle routing problems, Liu [45] first introduced uncertain programming to model vehicle routing problems with uncertain travel times. After that, Dong-Wang [14] proposed a new uncertain model for the vehicle routing problem with uncertain factors. Furthermore, Li-Peng-Li-Su [40] investigated a problem of dispatching medical supplies in emergency events, in which both demands and running times were assumed to be uncertain variables. They introduced two novel uncertain models, and designed a hybrid intelligent algorithm for solving them in general cases.

For location problems, Gao [24] first investigated the single facility location problem. By introducing the concepts of satisfaction degrees for both vertices and the whole network, two uncertain programming models were proposed. By considering service quality, setup costs and operating costs as indeterminate quantities, Zhou-Li-Wang [79] introduced an uncertain multi-objective programming model for the fire station location problem. In addition, Wang-Yang [64] studied a hierarchical facility location problem for reverse logistics network design under uncertainty. Wen-Qin-Kang [65] investigated an uncertain facility location-allocation problem, and proposed an uncertain model with the objective of minimizing the total transportation cost via α-optimistic criterion. Wu-Peng [67] studied a problem of logistics distribution center location under uncertainty, and proposed a chance-constrained model. In order to model the p-hub center problem based on experts's belief, Gao-Qin [27] proposed a chance constrained programming

model, and designed a hybrid intelligent algorithm to solve the proposed model. Recently, Zhang-Peng-Li [77] studied the covering location problem from two aspects in an uncertain environment. First, they introduced an uncertain location set covering model for the covering problem with uncertain variables. Second, they proposed two uncertain models, namely, (α, β)-maximal covering location model and α-chance maximal covering location model for the maximal covering problem with uncertain information. Chen-Peng-Zhang-Li [6] proposed an uncertain model for uncertain weight vertex covering problem. Furthermore, due to the increased environmental pressure, Zhang-Li-Li-Peng [78] investigated a sustainable multi-depot emergency facilities location-routing problem, and proposed a multi-objective uncertain model, in which the travel distances, demands and costs were assume to be uncertain variables.

For more detail of uncertain network optimization, the interested readers may refer to Peng-Zhang-Li [55].

Chapter 1

Graph and Network

It is quite well known that it is particularly convenient to describe the specified relationship between pairs of certain given objects by means of a graph. In fact, graphs can be used to model many types of relations in social systems. In this chapter, we will introduce some basic concepts of a graph used in the remaining parts of the book, including path, walk, cycle, diameter and connection, as well as the basic results closely related to these concepts. At the end of this chapter we will introduce some applications of the graph to network optimization problems.

1.1 Graph

Roughly speaking, a graph contains a set of vertices and a set of edges. A formal definition is given as follows.

Definition 1.1 *(Bondy-Murty [4]) A graph G is an ordered triple $(V(G), E(G), \psi_G)$ consisting of a nonempty set $V(G)$ of vertices, a set of $E(G)$, disjoint from $V(G)$, of edges and an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessarily distinct) vertices of G.*

An example graph should serve to clarity the definition.

Example 1.1: Let $G = (V(G), E(G), \psi_G)$ be a graph that is presented in Figure 1.1, where

$$V(G) = \{A, B, C, D, E\}$$

$$E(G) = \{e_1, e_2, e_3, e_4, e_5\}$$

and ψ_G is defined by

$$\psi_G(e_1) = AD, \quad \psi_G(e_2) = AB, \quad \psi_G(e_3) = BC$$

$$\psi_G(e_4) = CD, \quad \psi_G(e_5) = DE.$$

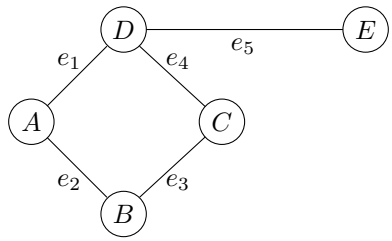


Figure 1.1: Graph G

Usually, a graph $G = (V(G), E(G), \psi_G)$ can be denoted as $G = (V(G), E(G))$ for short. If an edge connects a vertex to itself, it is called a loop. Two edges are called parallel edges if they have the same pair of vertices. A graph is simple if it has no loops and no parallel edges. And a graph is finite if both its vertex set and edge set are finite.

The number of vertices in graph G is often called the order of G, while the number of edges is called its size. To any simple finite graph G with n vertices, there corresponds an $n \times n$ matrix $A(G)$ called the adjacency matrix of G.

$$A(G) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

where

$$a_{ij} = \begin{cases} 1, & \text{if there exists an edge between vertices } v_i \text{ and } v_j \\ 0, & \text{otherwise,} \end{cases}$$

$i, j = 1, 2, \ldots, n$, respectively. The adjacency matrix of the graph that presented in Figure 1.1 is

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Generally speaking, $a_{ii} = 0$ for $i = 1, 2, \ldots, n$, and $\alpha_{ij} = \alpha_{ji}$ for any i and j since the graph is finite and simple.

We call a finite non-null sequence $W = v_0 e_1 v_1 e_2 v_2 \ldots e_k v_k$ is a *walk*, whose terms are alternately vertices and edges. The integer k is called the *length* of W. In a simple graph, a walk can be determined by its vertices completely. Hence W can be denoted as $v_0 v_1 \ldots v_k$ for short. If the edges of the walk W are distinct, W is called a *trail*. In addition, if the vertices of the

walk W are distinct, then W is called a *path*. Furthermore, if W has positive length and its origin and terminus are the same, W is *closed*. Naturally, a cycle is a closed trail whose origin and internal vertices are distinct.

Example 1.2: Figure 1.2 illustrates a walk, a trail, a path, a closed trail and a cycle in a graph.

Walk : $Ae_1Be_4Ce_7Fe_6De_3Be_4Ce_8E$
Trail : $Ae_2Ce_7Fe_6De_3Be_4Ce_8E$
Path : $Ae_2Ce_7Fe_6De_5G$
Closed trail : $Ee_8Ce_2Ae_1Be_4Ce_9E$
Cycle : $Be_4Ce_7Fe_6De_3B$

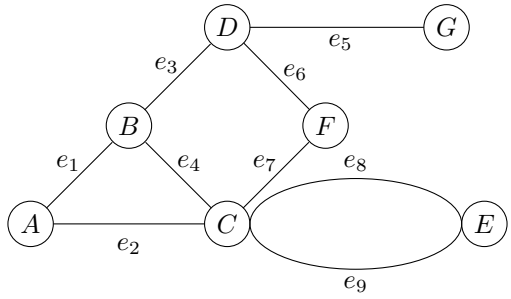


Figure 1.2: Graph G for Example 1.2

A graph H is called a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and ψ_H is the restriction of ψ_G to $E(H)$. A subgraph H with $V(H) = V(G)$ is called a spanning subgraph of G. In Figure 1.2, let $V(H_1) = \{A, B, C\}$, and $E(H_1) = \{e_1, e_2\}$, then H_1 is a subgraph of G. In addition, let $V(H_2) = \{A, B, C, D, E, F, G\}$, and $E(H_2) = \{e_1, e_2\}$, then H_2 is a spanning subgraph of G.

Let V' be a nonempty subset of $V(G)$. The *induced subgraph* by V' is a subgraph of G whose vertex set is V' and whose edge set is the set of those edges of G that have both ends in V'. Such subgraph is denoted by $G[V']$. $G[V \setminus V']$ is the subgraph obtained from G by deleting the vertices in V' together with their incident edges and is denoted by $G - V'$. We write $G - v$ for short if $V' = \{v\}$.

Let $u, v \in G$, u and v are said to be *connected* if there is a uv-path in G. It is easy to verify that connection is an equivalence relation on the vertex set $V(G)$. Let $V_1, V_2, \ldots, V_\omega$ be an equivalence partition of $V(G)$, two vertices u and v are connected if and only if they are in the same equivalent class V_i. The subgraphs $G[V_1], G[V_2], \ldots, G[V_\omega]$ are said to be the *connected component* of G. The number of connected components of G is denoted by $\omega(G)$. Then G is called to be *connected* if $\omega(G) = 1$.

In the following, we will introduce some special classes of graphs. A *complete graph* is a simple graph in which each pair of distinct vertices is

joined by an edge. If the vertex set of a graph can be partitioned into two subsets X and Y so that each edge has one vertex in X and one vertex in Y, then such graph is called a *bipartite graph*, and the partition (X, Y) is called a *bipartition* of the graph. Let G be a simple bipartite graph with bipartition (X, Y). If each vertex of X is joined by one edge to each vertex of Y, then G is said to be a *complete bipartite graph*. A subset M of $E(G)$ is called a *matching* in G if no two of elements of M are adjacent in G; The two end-vertices of an edge in M are said to be *matched under* M. A vertex x is *M-saturated* if some edge of M is incident with x; Otherwise, x is *M-unsaturated*. A matching M is *perfect* if every vertex of G is *M-saturated*. If G has no matching M' with $|M'| > |M|$, M is called a *maximum matching*. Obviously, every perfect matching is maximum.

1.2 Network

Let $G = (V, E)$ be a graph with each edge e of G associated to real number $w(e)$ called its weight. Then G, together with these weights on its edges, is called a weighted graph. A weighted graph is also called a network, which is denoted by $N = (V, E, w)$.

Let $N = (V, E, w)$ be a network. If $G_1 = (V_1, E_1)$ is a subgraph of $G(V, E)$, then $N_1 = (V_1, E_1, w)$ is called a subnetwork of N, and

$$w(N_1) = \sum_{e \in E_1} w(e)$$

is called the weight of N_1.

A network $N = (V, E, w)$, for example, is given in Figure 1.3. Let $G_1 = (V_1, E_1)$, where $V_1 = \{B, C, D\}$, $E_1 = \{BC, BD, CD\}$. Then G_1 is a subgraph of $G = (V, E)$. And $N_1 = (V_1, E_1, w')$, where $w' = \{w_2, w_4, w_5\}$, is a subnetwork of N.

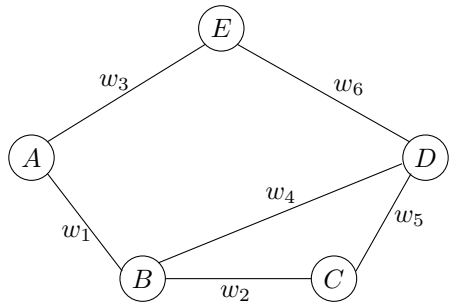


Figure 1.3: Network $N = (V, E, w)$

1.3 Network Optimization

Generally speaking, classical combinatorial optimization is the study of selecting an optimal object from a set of finite discrete objects. Network optimization problems are the combinatorial optimization problems associated with graphs and networks. Many optimization problems amount to finding a subgraph of a certain type in a weighted graph with minimum (or maximum) weight.

The Shortest Path Problem

The *shortest path problem* concentrates on finding the shortest route between two specified vertices in a network. For example, in a railway system, we want to know the shortest route between two specified cities in the network.

This railway system can be denoted by a network $N = (V, E, w)$, where the vertices represent the cities, the edges represent the rails and the weight $w(xy)$ of the edge xy represents the distance between two cities x and y.

The Chinese Postman Problem

A postman, he picks up mail at the post office, delivers it, and then ends his day back at the post office. Of course, he must travel each street in his area at least once. How can the postman choose his route such that he walks as little as possible. This problem is called the *Chinese postman problem*, since it was first studied by a Chinese mathematician, Kuan [38].

We can consider the Chinese postman problem on a network $N = (V, E, w)$, whose vertices represent the intersections of the streets, the edges represent the streets, and the weights of the edges represent the distances between two intersections. The weight of a tour $v_0e_1v_1 \dots e_nv_0$ is defined as

$$\sum_{i=1}^{n} w(e_i).$$

Clearly, the Chinese postman problem is to find a minimum-weight postman tour.

The Optimal Assignment Problem

Assume that in a certain company, m workers are variable for n jobs, each worker being qualified for one or more of these jobs. By adding virtual jobs or workers with 0 profitability, we may assume that the number of the workers and the jobs are equal, i.e, $m = n$. In an assignment, all the men are assigned, one man per job, to the jobs. One is interested in an assignment that maximises the total profit of the workers. The *optimal assignment problem* is just to find such an assignment.

We construct a weighted complete bipartite graph with bipartition (X, Y), where $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$, and edge $x_i y_j$ has weight w_{ij}, the profit of worker X_i in job Y_j. The optimal assignment problem becomes the problem of finding a maximum-weight perfect matching in this weighted graph.

Chapter 2

Uncertainty Theory

Uncertainty theory was founded by Liu [44] in 2007 and perfected by Liu [48] in 2015 to handle belief degrees based on normality, duality, subadditivity and product axioms. Nowadays, uncertainty theory has been successfully applied to many fields such as uncertain programming, uncertain risk analysis, uncertain reliability analysis, uncertain inference, uncertain finance, and so on. The emphases of this chapter is to introduce some necessary preliminary concepts and results selected from uncertainty theory.

2.1 Uncertain Measure

Let Γ be a nonempty set. A collection $\mathcal{L}$ of subsets of Γ is said to be a σ-algebra if it satisfies the following conditions: (a) $\Gamma \in \mathcal{L}$; (b) if $\Lambda \in \mathcal{L}$, then $\Lambda^c \in \mathcal{L}$; and (c) if $\Lambda_1, \Lambda_2, \cdots \in \mathcal{L}$, then $\bigcup_{i=1}^{\infty} \Lambda_i \in \mathcal{L}$.

Example 2.1: The collection $\{\emptyset, \Gamma\}$ is the smallest σ-algebra over Γ, and the power set is the largest σ-algebra.

Let $\mathcal{L}$ be a σ-algebra over nonempty set Γ. Each element Λ of $\mathcal{L}$ is called an event. Uncertain measure $\mathcal{M}$ is a function from $\mathcal{L}$ to $[0, 1]$. In order to rationally deal with belief degrees, Liu [44] proposed the following three axioms that an uncertain measure is supposed to satisfy:

Axiom 1. *(Normality Axiom)* $\mathcal{M}\{\Gamma\} = 1$ *for the universal set* Γ.

Axiom 2. *(Duality Axiom)* $\mathcal{M}\{\Lambda\} + \mathcal{M}\{\Lambda^c\} = 1$ *for any event* Λ.

Axiom 3. *(Subadditivity Axiom) For every countable sequence of events* $\{\Lambda_i\}$*, we have*

$$\mathcal{M}\left\{\bigcup_{i=1}^{\infty} \Lambda_i\right\} \leq \sum_{i=1}^{\infty} \mathcal{M}\{\Lambda_i\}.$$

In this case, the triple $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be an uncertainty space. Accordingly, the product axiom was defined by Liu [46] to obtain an uncertain measure of compound event.

Axiom 4. *(Product Axiom) Let $(\Gamma_k, \mathcal{L}_k, \mathcal{M}_k)$ be uncertainty spaces for $k = 1, 2, \ldots$. The product uncertain measure $\mathcal{M}$ on the product σ-algebra $\mathcal{L}_1 \times \mathcal{L}_2 \times \cdots$ is an uncertain measure satisfying*

$$\mathcal{M}\left\{\prod_{k=1}^{\infty} \Lambda_k\right\} = \bigwedge_{k=1}^{\infty} \mathcal{M}_k\{\Lambda_k\} \tag{2.1}$$

where Λ_k are arbitrarily chosen events from $\mathcal{L}_k$ for $k = 1, 2, \ldots$, respectively.

Example 2.2: Let $\mathcal{L}$ be the power set of $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$. Define

$$\mathcal{M}\{\gamma_1\} = 0.3, \quad \mathcal{M}\{\gamma_2\} = 0.5, \quad \mathcal{M}\{\gamma_3\} = 0.4,$$

$$\mathcal{M}\{\gamma_1, \gamma_2\} = 0.6, \quad \mathcal{M}\{\gamma_1, \gamma_3\} = 0.5, \quad \mathcal{M}\{\gamma_2, \gamma_3\} = 0.7,$$

$$\mathcal{M}\{\emptyset\} = 0, \quad \mathcal{M}\{\Gamma\} = 1.$$

It is easy to verify that $\mathcal{M}$ is an uncertain measure, and $(\Gamma, \mathcal{L}, \mathcal{M})$ is an uncertainty space.

Theorem 2.1 *(Liu [47], Monotonicity Theorem) Let Λ_1, Λ_2 be any two events of an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ with $\Lambda_1 \subset \Lambda_2$. Then we have*

$$\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}.$$

Proof: Obviously, we have $\Gamma = \Lambda_1^c \cup \Lambda_2$ since $\Lambda_1 \subset \Lambda_2$. Additionally, it follows from the subadditivity axiom that

$$1 = \mathcal{M}\{\Gamma\} \leq \mathcal{M}\{\Lambda_1^c\} + \mathcal{M}\{\Lambda_2\} = 1 - \mathcal{M}\{\Lambda_1\} + \mathcal{M}\{\Lambda_2\}.$$

Thus $\mathcal{M}\{\Lambda_1\} \leq \mathcal{M}\{\Lambda_2\}$. The theorem is proved.

Theorem 2.2 *(Liu [44]) Let $\mathcal{M}$ be an uncertain measure. Then the empty set $\emptyset$ has an uncertain measure zero, i.e.,*

$$\mathcal{M}\{\emptyset\} = 0.$$

Proof: It is clear that $\emptyset = \Gamma^c$ and $\mathcal{M}\{\Gamma\} = 1$. By using the duality axiom, we have

$$\mathcal{M}\{\emptyset\} = 1 - \mathcal{M}\{\Gamma\} = 1 - 1 = 0.$$

2.2 Uncertain Variable and Its Distribution

Definition 2.1 *(Liu [44]) An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set*

$$\{\xi \in B\} = \{\gamma \in \Gamma | \xi(\gamma) \in B\}$$

is an event.

Example 2.3: For the uncertain space defined in Example 2.2, the function

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2 \\ 1, & \text{if } \gamma = \gamma_3 \end{cases}$$

is an uncertain variable.

Usually, a function is said to be Boolean if it is a mapping from $\{0,1\}^n$ to $\{0,1\}$. In the same way, an uncertain variable is said to be Boolean if it takes values either 0 or 1 with uncertain measures. That is, a Boolean uncertain variable can be described as follows,

$$\xi = \begin{cases} 1, & \text{with uncertain measure } \alpha \\ 0, & \text{with uncertain measure } 1-\alpha \end{cases}$$

where $\alpha \in [0,1]$.

Theorem 2.1 *(Liu [44]) Let f be a real-valued measurable function, and let $\xi_1, \xi_2, \ldots, \xi_n$ be uncertain variables on $(\Gamma, \mathcal{L}, \mathcal{M})$. Then $f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable.*

Proof: According to Definition 2.1, $\xi_1, \xi_2, \ldots, \xi_n$ are measurable functions from $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. Thus $f(\xi_1, \xi_2, \ldots, \xi_n)$ is also a measurable function from $(\Gamma, \mathcal{L}, \mathcal{M})$ to the set of real numbers. That is, $f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable. The proof is completed.

Definition 2.2 *(Liu [46]) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if*

$$\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Theorem 2.2 *(Liu [46]) The uncertain variables $\xi_1, \xi_2, \ldots, \xi_n$ are independent if and only if*

$$\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in B_i)\right\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\} \tag{2.2}$$

for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers.

Proof: Assume that $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables. By using the duality axiom, we have

$$\begin{aligned}\mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in B_i)\right\} &= 1 - \mathcal{M}\left\{\left(\bigcup_{i=1}^{n}(\xi_i \in B_i)\right)^c\right\} \\ &= 1 - \mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i^c)\right\} \\ &= 1 - \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i^c\} = \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}.\end{aligned}$$

Thus the equation (2.2) holds. Conversely, if the equation (2.2) holds for any Borel sets $B_1, B_2, \ldots, B_n$ of real numbers, then we have

$$\begin{aligned}\mathcal{M}\left\{\bigcap_{i=1}^{n}(\xi_i \in B_i)\right\} &= 1 - \mathcal{M}\left\{\left(\bigcap_{i=1}^{n}(\xi_i \in B_i)\right)^c\right\} \\ &= 1 - \mathcal{M}\left\{\bigcup_{i=1}^{n}(\xi_i \in B_i^c)\right\} \\ &= 1 - \bigvee_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i^c\} = \bigwedge_{i=1}^{n} \mathcal{M}\{\xi_i \in B_i\}.\end{aligned}$$

It shows that $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables. Thus the theorem is proved.

Definition 2.3 *(Liu [44]) Let ξ be an uncertain variable. The uncertainty distribution Φ of ξ is defined as*

$$\Phi(x) = \mathcal{M}\{\xi \leq x\}$$

for any real number x.

Example 2.4: Take an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ to be $\{\gamma_1, \gamma_2\}$ with power set and $\mathcal{M}\{\gamma_1\} = 0.4$ and $\mathcal{M}\{\gamma_2\} = 0.6$. The uncertain variable ξ is defined as follows,

$$\xi(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2. \end{cases}$$

It is easy to verify that ξ has an uncertainty distribution

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 0 \\ 0.6, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } 1 \leq x. \end{cases}$$

Example 2.5: Consider an uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ with $\Gamma = \{\gamma_1, \gamma_2, \gamma_3\}$ and

$$\mathcal{M}\{\gamma_1\} = 0.3, \quad \mathcal{M}\{\gamma_2\} = 0.5, \quad \mathcal{M}\{\gamma_3\} = 0.3.$$

The uncertain variable ξ is defined as follows,

$$\xi(\gamma) = \begin{cases} -2, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2 \\ 2, & \text{if } \gamma = \gamma_3. \end{cases}$$

Then, the uncertainty distribution of ξ is

$$\Phi(x) = \begin{cases} 0, & \text{if } x < -2 \\ 0.3, & \text{if } -2 \leq x < 0 \\ 0.7, & \text{if } 0 \leq x < 2 \\ 1, & \text{if } 2 \leq x. \end{cases}$$

Definition 2.4 *(Liu [44]) Uncertain variables are said to be identically distributed if they have a common uncertainty distribution.*

Obviously, if uncertain variables $\xi = \eta$, then they are identically distributed. On the contrary, identical distribution does not mean that $\xi = \eta$.

Example 2.6: Let $(\Gamma, \mathcal{L}, \mathcal{M})$ be $\{\gamma_1, \gamma_2, \gamma_3\}$ with $\mathcal{M}\{\gamma_1\} = 0.3$, $\mathcal{M}\{\gamma_2\} = 0.6$, $\mathcal{M}\{\gamma_3\} = 0.3$. Define

$$\xi(\gamma) = \begin{cases} -1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2 \\ 1, & \text{if } \gamma = \gamma_3, \end{cases} \qquad \eta(\gamma) = \begin{cases} 1, & \text{if } \gamma = \gamma_1 \\ 0, & \text{if } \gamma = \gamma_2 \\ -1, & \text{if } \gamma = \gamma_3. \end{cases}$$

Then ξ and η have the same uncertainty distribution,

$$\Phi(x) = \begin{cases} 0, & \text{if } x < -1 \\ 0.3, & \text{if } -1 \leq x < 0 \\ 0.7, & \text{if } 0 \leq x < 1 \\ 1, & \text{if } x \geq 1. \end{cases}$$

It is clear that the two uncertain variables ξ and η are identically distributed but $\xi \neq \eta$.

In addition, Peng-Iwamura [56] and Liu-Lio [49] proved that a real-valued function $\Phi(x) : \Re \to [0, 1]$ is an uncertainty distribution if and only if it is a monotone increasing function satisfying $0 \leq \Phi(x) \leq 1$, $\Phi(x) \not\equiv 0$, $\Phi(x) \not\equiv 1$, and $\Phi(x_0) = 1$ if $\Phi(x) = 1$ for any $x > x_0$.

Some Uncertainty Distributions

Definition 2.5 *(Liu [47]) If an uncertain variable ξ has a linear uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ (x-a)/(b-a), & \text{if } a \leq x < b \\ 1, & \text{if } x \geq b \end{cases}$$

where a and b are real numbers with $a < b$, we call it linear and denote this by $\xi \sim \mathcal{L}(a, b)$.

Definition 2.6 *(Liu [47]) If an uncertain variable ξ has a zigzag uncertainty distribution*

$$\Phi(x) = \begin{cases} 0, & \text{if } x < a \\ (x-a)/2(b-a), & \text{if } a \leq x < b \\ (x+c-2b)/2(c-b), & \text{if } b \leq x < c \\ 1, & \text{if } x \geq c \end{cases}$$

where a, b, c are real numbers with $a < b < c$, we call it zigzag and denote this by $\xi \sim \mathcal{Z}(a, b, c)$.

Definition 2.7 *(Liu [47]) If an uncertain variable ξ has a normal uncertainty distribution*

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \in \Re$$

where e and σ are real numbers with $\sigma > 0$, we call it normal and denote this by $\xi \sim \mathcal{N}(e, \sigma)$.

Definition 2.8 *(Liu [47]) If an uncertain variable ξ has a lognormal uncertainty distribution*

$$\Phi(x) = \left(1 + \exp\left(\frac{\pi(e-\ln x)}{\sqrt{3}\sigma}\right)\right)^{-1}, \quad x \geq 0$$

where e and σ are real numbers with $\sigma > 0$, we call it lognormal and denote this by $\xi \sim \mathcal{LOGN}(e, \sigma)$.

Theorem 2.3 *(Liu [47]) Let ξ be an uncertain variable with uncertainty distribution Φ. Then for any real number x, we have*

$$\mathcal{M}\{\xi \leq x\} = \Phi(x), \quad \mathcal{M}\{\xi > x\} = 1 - \Phi(x). \tag{2.3}$$

Proof: It follows from the definition of uncertainty distribution that we obtain the equation

$$\mathcal{M}\{\xi \leq x\} = \Phi(x)$$

immediately. In addition, due to the duality of uncertain measure, the following result

$$\mathcal{M}\{\xi > x\} = 1 - \mathcal{M}\{\xi \leq x\} = 1 - \Phi(x)$$

holds. The proof is completed.

Remark 2.1: (Liu [48]) When the uncertainty distribution Φ is a continuous function, we also have

$$\mathcal{M}\{\xi < x\} = \Phi(x), \quad \mathcal{M}\{\xi \geq x\} = 1 - \Phi(x). \tag{2.4}$$

Definition 2.9 *(Liu [47]) An uncertainty distribution $\Phi(x)$ is called regular if it is a continuous and strictly increasing function with respect to x at which $0 < \Phi(x) < 1$, and*

$$\lim_{x \to -\infty} \Phi(x) = 0, \quad \lim_{x \to +\infty} \Phi(x) = 1. \tag{2.5}$$

It is easy to verify that the linear uncertainty distribution, zigzag uncertainty distribution, normal uncertainty distribution, and lognormal uncertainty distribution are all regular.

Inverse Uncertainty Distribution

Definition 2.10 *(Liu [47]) Let ξ be an uncertain variable with a regular uncertainty distribution $\Phi(x)$. Then the inverse function $\Phi^{-1}(\alpha)$ on the open interval $(0, 1)$ is said to be the inverse uncertainty distribution of ξ.*

Example 2.7: (Liu [47]) The linear uncertain variable $\mathcal{L}(a, b)$ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = (1 - \alpha)a + \alpha b.$$

Example 2.8: (Liu [47]) The zigzag uncertain variable $\mathcal{Z}(a, b, c)$ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = \begin{cases} (1 - 2\alpha)a + 2\alpha b, & \text{if } \alpha < 0.5 \\ (2 - 2\alpha)b + (2\alpha - 1)c, & \text{if } \alpha \geq 0.5. \end{cases}$$

Example 2.9: (Liu [47]) The normal uncertain variable $\mathcal{N}(e, \sigma)$ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}.$$

Example 2.10: (Liu [47]) The lognormal uncertain variable $\mathcal{LOGN}(e, \sigma)$ has an inverse uncertainty distribution

$$\Phi^{-1}(\alpha) = \exp\left(e + \frac{\sigma\sqrt{3}}{\pi} \ln \frac{\alpha}{1 - \alpha}\right).$$

2.3 Operational Law

A real-valued function $f(x_1, x_2, \ldots, x_n)$ is said to be strictly monotone if it is strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$. Note that both the strictly increasing function and strictly decreasing function are special cases of the strictly monotone function. The following two functions are strictly monotone functions,

$$f(x_1, x_2, x_3) = x_1 - x_2 + x_3,$$

$$f(x_1, x_2) = \frac{x_1}{x_2}, \qquad x_1, x_2 > 0.$$

The following two functions are strictly increasing functions,

$$f(x_1, x_2, x_3) = x_1 + x_2 + x_3,$$

$$f(x_1, x_2) = x_1 x_2, \qquad x_1, x_2 > 0.$$

The following two functions are strictly decreasing functions,

$$f(x_1, x_2, x_3) = -x_1 - x_2 - x_3,$$

$$f(x_1, x_2) = \frac{1}{x_1 x_2}, \qquad x_1, x_2 > 0.$$

Theorem 2.4 *(Liu [47]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. Assume the function $f(x_1, x_2, \ldots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$. Then the uncertain variable*

$$\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$$

has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

Proof: For the sake of simplicity, we only prove the case of $m = 1$ and $n = 2$. Since f is a strictly monotone function, we have

$$\{f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1-\alpha))\} \supset \{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \cap \{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\}.$$

According to Theorem 2.1 and independence of ξ_1 and ξ_2, we get

$$\begin{aligned} & \mathcal{M}\{f(\xi_1, \xi_2) \leq f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(1-\alpha))\} \\ \geq\ & \mathcal{M}\{(\xi_1 \leq \Phi_1^{-1}(\alpha)) \cap (\xi_2 \geq \Phi_2^{-1}(1-\alpha))\} \\ =\ & \mathcal{M}\{\xi_1 \leq \Phi_1^{-1}(\alpha)\} \wedge \mathcal{M}\{\xi_2 \geq \Phi_2^{-1}(1-\alpha)\} \\ =\ & \alpha \wedge \alpha = \alpha. \end{aligned}$$

Namely,

$$\mathcal{M}\{f(\xi_1,\xi_2) \le f(\Phi_1^{-1}(\alpha),\Phi_2^{-1}(1-\alpha))\} \ge \alpha.$$

Additionally, since f is a strictly monotone function, we have

$$\{f(\xi_1,\xi_2) \le f(\Phi_1^{-1}(\alpha),\Phi_2^{-1}(1-\alpha))\} \subset \{\xi_1 \le \Phi_1^{-1}(\alpha)\} \cup \{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\}.$$

According to Theorem 2.1 and independence of ξ_1 and ξ_2, we get

$$\begin{aligned} & \mathcal{M}\{f(\xi_1,\xi_2) \le f(\Phi_1^{-1}(\alpha),\Phi_2^{-1}(1-\alpha))\} \\ \le\ & \mathcal{M}\{(\xi_1 \le \Phi_1^{-1}(\alpha)) \cup (\xi_2 \ge \Phi_2^{-1}(1-\alpha))\} \\ =\ & \mathcal{M}\{\xi_1 \le \Phi_1^{-1}(\alpha)\} \vee \mathcal{M}\{\xi_2 \ge \Phi_2^{-1}(1-\alpha)\} \\ =\ & \alpha \vee \alpha = \alpha. \end{aligned}$$

Namely,

$$\mathcal{M}\{f(\xi_1,\xi_2) \le f(\Phi_1^{-1}(\alpha),\Phi_2^{-1}(1-\alpha))\} \le \alpha.$$

Thus, we have $\mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} = \mathcal{M}\{f(\xi_1,\xi_2) \le f(\Phi_1^{-1}(\alpha),\Phi_2^{-1}(1-\alpha))\} = \alpha$. That is, Ψ^{-1} is just the inverse uncertainty distribution of ξ. The theorem is proved.

Remark 2.2: (Liu [47]) Let $\xi_1,\xi_2,\ldots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\ldots,\Phi_n$, respectively. If $f(x_1,x_2,\ldots,x_n)$ is a continuous and strictly increasing function, then $\xi = f(\xi_1,\xi_2,\ldots,\xi_n)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha),\Phi_2^{-1}(\alpha),\ldots,\Phi_n^{-1}(\alpha)).$$

Remark 2.3: (Liu [47]) Let $\xi_1,\xi_2,\ldots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\ldots,\Phi_n$, respectively. If $f(x_1,x_2,\ldots,x_n)$ is a continuous and strictly decreasing function, then $\xi = f(\xi_1,\xi_2,\ldots,\xi_n)$ has an inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(1-\alpha),\Phi_2^{-1}(1-\alpha),\ldots,\Phi_n^{-1}(1-\alpha)).$$

Example 2.11: Let ξ_1,ξ_2 and ξ_3 be independent uncertain variables with regular uncertainty distributions Φ_1,Φ_2 and Φ_3, respectively. Then the inverse uncertainty distribution of $\xi_1-\xi_2+\xi_3$ is

$$\Psi^{-1}(\alpha) = \Phi_1^{-1}(\alpha) - \Phi_2^{-1}(1-\alpha) + \Phi_3^{-1}(\alpha).$$

Example 2.12: Let ξ_1 and ξ_2 be independent and positive uncertain variables with regular uncertainty distributions Φ_1 and Φ_2, respectively. Then the inverse uncertainty distribution of $\exp(\xi_1/\xi_2)$ is

$$\Psi^{-1}(\alpha) = \exp\left(\frac{\Phi_1^{-1}(\alpha)}{\Phi_2^{-1}(1-\alpha)}\right),$$

and the inverse uncertainty distribution of $\exp(\xi_1 \times \xi_2)$ is

$$\Upsilon^{-1}(\alpha) = \exp\left(\Phi_1^{-1}(\alpha) \times \Phi_2^{-1}(\alpha)\right).$$

Theorem 2.5 *(Liu [47]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent Boolean uncertain variables. That is,*

$$\xi_i = \begin{cases} 1 \text{ with uncertain measure } a_i \\ 0 \text{ with uncertain measure } 1 - a_i \end{cases}$$

for $i = 1, 2, \ldots, n$. Assume f is a Boolean function (not necessarily monotone). Then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = \begin{cases} \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if} \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) < 0.5 \\ 1 - \sup\limits_{f(x_1,x_2,\ldots,x_n)=0} \min\limits_{1\le i\le n} \nu_i(x_i), \\ \qquad \text{if} \sup\limits_{f(x_1,x_2,\ldots,x_n)=1} \min\limits_{1\le i\le n} \nu_i(x_i) \ge 0.5 \end{cases} \tag{2.6}$$

where x_i take values either 0 or 1, and ν_i are defined by

$$\nu_i(x_i) = \begin{cases} a_i, & \text{if } x_i = 1 \\ 1 - a_i, & \text{if } x_i = 0 \end{cases}$$

for $i = 1, 2, \ldots, n$, respectively.

In addition, Gao-Qin [26] proved the following results for monotone functions. Interested readers may refer to Gao-Qin [26] for details.

Theorem 2.6 *(Gao-Qin [26]) Let $\xi_1, \xi_2, \ldots, \xi_m$ be independent uncertain variables, and $f : \Re^m \to \Re$ a monotone function. Assume*

$$\xi_i \in \Omega_i = \left\{b_1^{(i)}, b_2^{(i)}, \ldots, b_{h_i}^{(i)}\right\}, \quad i = 1, 2, \ldots, m$$

where $h_i \ge 2$ and $b_1^{(i)} < b_2^{(i)} < \cdots < b_{h_i}^{(i)}$. Then the distribution of $\eta = f(\xi_1, \xi_2, \ldots, \xi_n)$ can be obtained by

$$\mathcal{M}\{\eta \le k\} = \sup_{f(B_1,B_2,\ldots,B_m)\subset(-\infty,k]} \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\}$$

where B_i is the subset of $\{b_1^{(i)}, b_2^{(i)}, \ldots, b_{h_i}^{(i)}\}$, $i = 1, 2, \ldots, m$.

Letting $\Omega_1 = \Omega_2 = \cdots = \Omega_m = \{0, 1\}$, it directly follows from Theorem 2.6 that, we have

Theorem 2.7 *(Gao-Qin [26]) Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} 1 \text{ with uncertain measure } a_i \\ 0 \text{ with uncertain measure } 1 - a_i \end{cases}$$

for $i = 1, 2, \ldots, n$. If f is a monotone function, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is an uncertain variable such that

$$\mathcal{M}\{\xi \geq k\} = \sup_{f(B_1, B_2, \ldots, B_n) \subset [k, +\infty)} \min_{1 \leq i \leq n} \mathcal{M}\{\xi_i \in B_i\}$$

where B_i is the subset of $\{0, 1\}$, $i = 1, 2, \ldots, n$.

For an increasing Boolean function, Gao-Gao [22] provided a more precisely result in the following way,

Theorem 2.8 *(Gao-Gao [22]) Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent Boolean uncertain variables, i.e.,*

$$\xi_i = \begin{cases} 1 \text{ with uncertain measure } a_i \\ 0 \text{ with uncertain measure } 1 - a_i \end{cases} \tag{2.7}$$

for $i = 1, 2, \ldots, n$. If f is an increasing Boolean function, then $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ is a Boolean uncertain variable such that

$$\mathcal{M}\{\xi = 1\} = \sup_{f(B_1, B_2, \ldots, B_n) = 1} \min_{1 \leq i \leq n} \mathcal{M}\{\xi_i \in B_i\} \tag{2.8}$$

where B_i are subsets of $\{0, 1\}$, $i = 1, 2, \ldots, n$.

2.4 Expected Value

The expected value, which represents the average value of an uncertain variable, is a frequently employed criterion to rank uncertain variables.

Definition 2.11 *(Liu [44]) Let ξ be an uncertain variable. Then the expected value of ξ is defined by*

$$E[\xi] = \int_0^{+\infty} \mathcal{M}\{\xi \geq x\} \mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\} \mathrm{d}x$$

provided that at least one of the two integrals is finite.

Theorem 2.9 *(Liu [44]) If an uncertain variable ξ with an uncertainty distribution Φ has an expected value $E[\xi]$, then*

$$E[\xi] = \int_0^{+\infty} (1 - \Phi(x)) \mathrm{d}x - \int_{-\infty}^0 \Phi(x) \mathrm{d}x.$$

Proof: For almost all numbers x, the measure inversion theorem shows that $\mathcal{M}\{\xi \geq x\} = 1 - \Phi(x)$ and $\mathcal{M}\{\xi \leq x\} = \Phi(x)$. According to the definition of the expected value operator, we have

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} \mathcal{M}\{\xi \geq x\}\mathrm{d}x - \int_{-\infty}^0 \mathcal{M}\{\xi \leq x\}\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x. \end{aligned}$$

The theorem is proved.

Theorem 2.10 *(Liu [47]) Let ξ be an uncertain variable with regular uncertainty distribution Φ. If the expected value $E[\xi]$ exists, then*

$$E[\xi] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha. \tag{2.9}$$

Proof: It follows from the integration by parts and Theorem 2.9 that the expected value is

$$\begin{aligned} E[\xi] &= \int_0^{+\infty} (1 - \Phi(x))\mathrm{d}x - \int_{-\infty}^0 \Phi(x)\mathrm{d}x \\ &= \int_0^{+\infty} x\mathrm{d}\Phi(x) + \int_{-\infty}^0 x\mathrm{d}\Phi(x) \\ &= \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x). \end{aligned}$$

Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, then the expected value is

$$E[\xi] = \int_{-\infty}^{+\infty} x\mathrm{d}\Phi(x) = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha.$$

The theorem is proved.

Example 2.13: Let ξ be a linear uncertain variable $\mathcal{L}(a, b)$, then it has an expected value

$$E[\xi] = \frac{a+b}{2}.$$

Example 2.14: Let ξ be a zigzag uncertain variable $\mathcal{Z}(a, b, c)$, then it has an expected value

$$E[\xi] = \frac{a+2b+c}{4}.$$

Example 2.15: Let ξ be a normal uncertain variable $\mathcal{N}(e, \sigma)$, then it has an expected value e, i.e.,

$$E[\xi] = e.$$

Theorem 2.11 *(Liu-Ha [50]) Assume $\xi_1, \xi_2, \ldots, \xi_n$ are independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \ldots, x_m$ and strictly decreasing with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then the uncertain variable $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$ has an expected value*

$$E[\xi] = \int_0^1 f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha. \quad (2.10)$$

Proof: According to Theorem 2.4, the inverse uncertainty distribution of ξ is

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha)).$$

It follows from Theorem 2.10 that equation (2.10) holds immediately. The theorem is proved.

Example 2.16: Let ξ and η be independent uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. Then

$$E[\xi + \eta] = \int_0^1 \left(\Phi^{-1}(\alpha) + \Psi^{-1}(\alpha)\right) \mathrm{d}\alpha,$$

and

$$E[\xi - \eta] = \int_0^1 \left(\Phi^{-1}(\alpha) - \Psi^{-1}(1-\alpha)\right) \mathrm{d}\alpha.$$

Theorem 2.12 *(Liu [47]) Let ξ and η be independent uncertain variables with regular uncertainty distributions Φ and Ψ, respectively. If the expected values $E[\xi]$ and $E[\eta]$ exist, then for any real numbers a and b, we have*

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta]. \quad (2.11)$$

Proof: We will prove the theorem via the following three steps.

Step 1: We first prove $E[a\xi] = aE[\xi]$ for any real number a. Clearly, the equation holds if $a = 0$. If $a > 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(\alpha).$$

Theorem 2.10 shows that

$$E[a\xi] = \int_0^1 a\Phi^{-1}(\alpha)\mathrm{d}\alpha = a\int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha = aE[\xi].$$

If $a < 0$, then the inverse uncertainty distribution of $a\xi$ is

$$\Upsilon^{-1}(\alpha) = a\Phi^{-1}(1-\alpha).$$

Theorem 2.10 shows that

$$E[a\xi] = \int_0^1 a\Phi^{-1}(1-\alpha)\mathrm{d}\alpha = a\int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha = aE[\xi].$$

Thus we always have $E[a\xi] = aE[\xi]$.

STEP 2: We prove $E[\xi+\eta] = E[\xi]+E[\eta]$. Since ξ and η are independent uncertain variables, then the inverse uncertainty distribution of $\xi+\eta$ is

$$\Phi^{-1}(\alpha) + \Psi^{-1}(\alpha).$$

Theorem 2.10 shows that

$$E[\xi+\eta] = \int_0^1 \Phi^{-1}(\alpha)\mathrm{d}\alpha + \int_0^1 \Psi^{-1}(\alpha)\mathrm{d}\alpha = E[\xi]+E[\eta].$$

STEP 3: Since $a\xi$ and $b\eta$ are independent uncertain variables for any real numbers a and b, then it follows from Steps 1 and 2 that

$$E[a\xi+b\eta] = E[a\xi]+E[b\eta] = aE[\xi]+bE[\eta].$$

The theorem is proved.

2.5 Variance

Definition 2.12 *(Liu [44]) If an uncertain variable ξ has a finite expected value e, then the variance of ξ is*

$$V[\xi] = E[(\xi-e)^2].$$

Remark 2.4: Since $(\xi-e)^2$ is a nonnegative uncertain variable, it follows from Definition 2.11 that

$$V[\xi] = \int_0^{+\infty} \mathcal{M}\{(\xi-e)^2 \geq x\}\mathrm{d}x.$$

Theorem 2.13 *(Liu [44]) Let ξ be an uncertain variable with finite expected value e. For any real numbers a and b, we have*

$$V[a\xi+b] = a^2V[\xi].$$

Proof: It is easy to verify that the expected value of $a\xi+b$ is $ae+b$. According to the definition of variance, we have

$$V[a\xi+b] = E\left[(a\xi+b-(ae+b))^2\right] = a^2E[(\xi-e)^2] = a^2V[\xi].$$

The proof is completed.

Let ξ be an uncertain variable with an uncertainty distribution Φ. If ξ has a finite expected value e, then an upper bound of the variance could be derived as follows,

$$\begin{aligned} V[\xi] &= \int_0^{+\infty} \mathcal{M}\{(\xi - e)^2 \geq x\}\mathrm{d}x \\ &= \int_0^{+\infty} \mathcal{M}\{(\xi \geq e + \sqrt{x}) \cup (\xi \leq e - \sqrt{x})\}\mathrm{d}x \\ &\leq \int_0^{+\infty} (\mathcal{M}\{\xi \geq e + \sqrt{x}\} + \mathcal{M}\{\xi \leq e - \sqrt{x}\})\mathrm{d}x \\ &= \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))\mathrm{d}x. \end{aligned}$$

Stipulation 2.1 *(Liu [47]) Let ξ be an uncertain variable with an uncertainty distribution Φ. If ξ has a finite expected value e, then*

$$V[\xi] = \int_0^{+\infty} (1 - \Phi(e + \sqrt{x}) + \Phi(e - \sqrt{x}))\mathrm{d}x. \tag{2.12}$$

Theorem 2.14 *(Liu [48]) Let ξ be an uncertain variable with an uncertainty distribution Φ. If ξ has a finite expected value e, then*

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 \mathrm{d}\Phi(x). \tag{2.13}$$

Proof: Based on Stipulation 2.1, the variance of ξ is

$$V[\xi] = \int_0^{+\infty} (1 - \Phi(e + \sqrt{y}))\mathrm{d}y + \int_0^{+\infty} \Phi(e - \sqrt{y})\mathrm{d}y.$$

Substituting $e + \sqrt{y}$ with x and y with $(x - e)^2$, it follows from the change of variables and integration by parts that

$$\int_0^{+\infty} (1 - \Phi(e + \sqrt{y}))\mathrm{d}y = \int_e^{+\infty} (1 - \Phi(x))\mathrm{d}(x - e)^2 = \int_e^{+\infty} (x - e)^2 \mathrm{d}\Phi(x).$$

In the same way, substituting $e - \sqrt{y}$ with x and y with $(x - e)^2$, we get

$$\int_0^{+\infty} \Phi(e - \sqrt{y})\mathrm{d}y = \int_e^{-\infty} \Phi(x)\mathrm{d}(x - e)^2 = \int_{-\infty}^{e} (x - e)^2 \mathrm{d}\Phi(x).$$

Thus, we ahve

$$V[\xi] = \int_{-\infty}^{+\infty} (x - e)^2 \mathrm{d}\Phi(x).$$

The theorem thus is verified.

Theorem 2.15 *(Yao [70]) Let ξ be an uncertain variable with a regular uncertainty distribution Φ. If ξ has a finite expected value e, then*

$$V[\xi] = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 \mathrm{d}\alpha. \tag{2.14}$$

Proof: Substituting $\Phi(x)$ with α and x with $\Phi^{-1}(\alpha)$, by using of Theorem 2.14 that the variance is

$$V[\xi] = \int_{-\infty}^{+\infty} (x-e)^2 \mathrm{d}\Phi(x) = \int_0^1 (\Phi^{-1}(\alpha) - e)^2 \mathrm{d}\alpha.$$

The theorem is proved.

Example 2.17: If ξ is a linear uncertain variable $\mathcal{L}(a,b)$, it has a variance

$$V[\xi] = \frac{(b-a)^2}{12}.$$

Example 2.18: If ξ is a zigzag uncertain variable $\mathcal{Z}(a,b,c)$, it has a variance

$$V[\xi] = \frac{(5a^2 + 4b^2 + 5c^2 - 4ab - 6ac - 4bc)}{48}.$$

Example 2.19: If ξ is a normal uncertain variable $\mathcal{N}(e,\sigma)$, it has a variance

$$V[\xi] = \sigma^2.$$

Chapter 3

Uncertain Programming

As a class of mathematical programming involving uncertain variables, uncertain programming was founded by Liu [45] in 2009. Uncertain programming has been used in different ways in the past. This chapter will provide the theory of uncertain programming, including expected value model, chance-constrained programming, and measure-chance programming.

Assume that $\boldsymbol{x}$ is a decision vector, $\boldsymbol{\xi}$ is an uncertain vector, $f(\boldsymbol{x}, \boldsymbol{\xi})$ is a objective function, and $g_j(\boldsymbol{x}, \boldsymbol{\xi})$ are constraint functions, $j = 1, 2, \ldots, p$. Let us consider the following "uncertain programming"

$$\begin{cases} \max\limits_{\boldsymbol{x}} f(\boldsymbol{x}, \boldsymbol{\xi}) \\ \text{subject to:} \\ \quad g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0, \quad j = 1, 2, \ldots, p. \end{cases} \tag{3.1}$$

Clearly, we cannot maximize the uncertain quantity $f(\boldsymbol{x}, \boldsymbol{\xi})$ directly. In addition, the uncertain constraints $g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0, j = 1, 2, \ldots, p$ do not produce a crisp feasible set. It tells us that the model (3.1) is not well-defined. To obtain an optimal decision in an uncertain environment with uncertain parameters, some additional criteria should be considered.

3.1 Expected Value Model

The first type of uncertain programming is the so-called expected value model (EVM), which attempts to optimize the expected value of the objective function subject to the expected constraints.

To obtain the optimal decision with the modeling idea of the expected

value, Liu [45] provided a spectrum of uncertain EVM

$$\begin{cases} \max\limits_{\boldsymbol{x}} E[f(\boldsymbol{x},\boldsymbol{\xi})] \\ \text{subject to:} \\ \quad E[g_j(\boldsymbol{x},\boldsymbol{\xi})] \leq 0, \quad j=1,2,\ldots,p. \end{cases} \tag{3.2}$$

Definition 3.1 *(Liu [45]) A vector $\boldsymbol{x}$ is said to be a feasible solution to the uncertain programming model (3.2) if and only if $E[g_j(\boldsymbol{x},\boldsymbol{\xi})] \leq 0$ for $j=1,2,\ldots,p$. A feasible solution $\boldsymbol{x}^*$ is said to be an optimal solution to the uncertain programming model (3.2) if*

$$E[f(\boldsymbol{x}^*,\boldsymbol{\xi})] \geq E[f(\boldsymbol{x},\boldsymbol{\xi})] \tag{3.3}$$

for any feasible solution $\boldsymbol{x}$.

Theorem 3.1 *(Liu [47]) Let $\xi_1,\xi_2,\ldots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\ldots,\Phi_n$, respectively. If the objective function $f(\boldsymbol{x},\xi_1,\xi_2,\ldots,\xi_n)$ is continuous, strictly increasing with respect to $\xi_1,\xi_2,\ldots,\xi_m$ and strictly decreasing with respect to $\xi_{m+1},\xi_{m+2},\ldots,\xi_n$, then $E[f(\boldsymbol{x},\xi_1,\xi_2,\ldots,\xi_n)]$ is equal to*

$$\int_0^1 f(\boldsymbol{x},\Phi_1^{-1}(\alpha),\ldots,\Phi_m^{-1}(\alpha),\Phi_{m+1}^{-1}(1-\alpha),\ldots,\Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha. \tag{3.4}$$

Proof: It follows from Theorem 2.11 immediately.

Theorem 3.2 *(Liu [47]) Let $\xi_1,\xi_2,\ldots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\ldots,\Phi_n$, respectively. If the constraint function $g(\boldsymbol{x},\xi_1,\xi_2,\ldots,\xi_n)$ is continuous, strictly increasing with respect to $\xi_1,\xi_2,\ldots,\xi_k$ and strictly decreasing with respect to $\xi_{k+1},\xi_{k+2},\ldots,\xi_n$, then the constraint*

$$E[g(\boldsymbol{x},\xi_1,\xi_2,\ldots,\xi_n)] \leq 0 \tag{3.5}$$

holds if and only if

$$\int_0^1 g(\boldsymbol{x},\Phi_1^{-1}(\alpha),\ldots,\Phi_k^{-1}(\alpha),\Phi_{k+1}^{-1}(1-\alpha),\ldots,\Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha \leq 0. \tag{3.6}$$

Proof: It follows from Theorem 2.11 immediately.

Theorem 3.3 *(Liu [47]) Let $\xi_1,\xi_2,\ldots,\xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1,\Phi_2,\ldots,\Phi_n$, respectively. If $f(\boldsymbol{x},\xi_1,\xi_2,\ldots,\xi_n)$ is continuous, strictly increasing with respect to $\xi_1,\xi_2,\ldots,\xi_m$ and strictly decreasing with respect to $\xi_{m+1},\xi_{m+2},\ldots,\xi_n$, and $g_j(\boldsymbol{x},\xi_1,\xi_2,\ldots,\xi_n)$ are continuous, strictly increasing with respect to $\xi_1,\xi_2,\ldots,\xi_k$ and*

strictly decreasing with respect to $\xi_{k+1}, \xi_{k+2}, \ldots, \xi_n$ for $j = 1, 2, \ldots, p$, then model (3.2) is equivalent to the crisp mathematical programming

$$\begin{cases} \max\limits_{\boldsymbol{x}} \displaystyle\int_0^1 f(\boldsymbol{x}, \Phi_1^{-1}(\alpha), \ldots, \Phi_m^{-1}(\alpha), \Phi_{m+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha \\ \text{subject to:} \\ \quad \displaystyle\int_0^1 g_j(\boldsymbol{x}, \Phi_1^{-1}(\alpha), \ldots, \Phi_k^{-1}(\alpha), \Phi_{k+1}^{-1}(1-\alpha), \ldots, \Phi_n^{-1}(1-\alpha))\mathrm{d}\alpha \leq 0 \\ \quad j = 1, 2, \ldots, p. \end{cases}$$

Proof: It follows from Theorems 3.1 and 3.2 immediately.

In many cases, a decision maker may consider multiple objectives simultaneously in a decision system. Suppose there are m objective functions. Thus Liu [45] presented the following uncertain expected value multiobjective programming (UEVMP),

$$\begin{cases} \max\limits_{\boldsymbol{x}} [E[f_1(\boldsymbol{x}, \boldsymbol{\xi})], E[f_2(\boldsymbol{x}, \boldsymbol{\xi})], \ldots, E[f_m(\boldsymbol{x}, \boldsymbol{\xi})]] \\ \text{subject to:} \\ \quad E[g_j(\boldsymbol{x}, \boldsymbol{\xi})] \leq 0, \quad j = 1, 2, \ldots, p \end{cases} \tag{3.7}$$

where $f_i(\boldsymbol{x}, \boldsymbol{\xi})$ are objective functions for $i = 1, 2, \ldots, m$, and $g_j(\boldsymbol{x}, \boldsymbol{\xi})$ are constraint functions for $j = 1, 2, \ldots, p$.

Definition 3.2 *(Liu [45]) A feasible solution $\boldsymbol{x}^*$ is called a Pareto solution to the UEVMP (3.7) if there is no feasible solution $\boldsymbol{x}$ such that*

$$E[f_i(\boldsymbol{x}, \boldsymbol{\xi})] \geq E[f_i(\boldsymbol{x}^*, \boldsymbol{\xi})], \qquad i = 1, 2, \ldots, m \tag{3.8}$$

and $E[f_j(\boldsymbol{x}, \boldsymbol{\xi})] > E[f_j(\boldsymbol{x}^*, \boldsymbol{\xi})]$ for at least one index j.

Goal programming is a powerful tool to deal with multiple objectives decision making. Assume that we have m goals and p constraints. According to the priority structure and target levels set by the decision maker, an expected value goal programming (EVGP) was formulated by Liu [45] as follows,

$$\begin{cases} \min\limits_{\boldsymbol{x}} \displaystyle\sum_{j=1}^{l} P_j \sum_{i=1}^{m} (u_{ij} d_i^+ + v_{ij} d_i^-) \\ \text{subject to:} \\ \quad E[f_i(\boldsymbol{x}, \boldsymbol{\xi})] + d_i^- - d_i^+ = b_i, \qquad i = 1, 2, \ldots, m \\ \quad E[g_j(\boldsymbol{x}, \boldsymbol{\xi})] \leq 0, \qquad j = 1, 2, \ldots, p \\ \quad d_i^-, d_i^+ \geq 0, \qquad i = 1, 2, \ldots, m \end{cases} \tag{3.9}$$

where l is the number of priorities, P_j are the preemptive priority factors such that $P_j \gg P_{j+1}$ for all j, u_{ij} and v_{ij} are the weighting factors corresponding

to positive deviation and negative deviation for goal i with priority j assigned, respectively, b_i is the target value corresponding to goal i, $d_i^+ = [E[f_i(\boldsymbol{x}, \boldsymbol{\xi})] - b_i] \vee 0$ is the positive deviation from the target of goal i and $d_i^- = [b_i - E[f_i(\boldsymbol{x}, \boldsymbol{\xi})]] \vee 0$ is the negative deviation from the target of goal i, f_i are the functions in goal constraints, and g_j are the functions in real constraints.

3.2 Chance-Constrained Programming

Chance-constrained programming, as the second type of mathematical programming developed by Charnes and Cooper [5], offers a powerful means of modeling a decision system with uncertain variables.

Assume that we have uncertain constraints $g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0, j = 1, 2, \ldots, p$, where $\boldsymbol{x}$ is a decision vector, and $\boldsymbol{\xi}$ is an uncertain vector. As mentioned before, since $g_j(\boldsymbol{x}, \boldsymbol{\xi})$ are uncertain variables, then the uncertain constraints do not define a deterministic feasible set. Instead, we hope that the uncertain constraints hold with a confidence level α. Then a chance constraint can be described as follows,

$$\mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0, j = 1, 2, \ldots, p\} \geq \alpha. \tag{3.10}$$

Definition 3.3 *(Liu [45]) A solution $\boldsymbol{x}$ is said to be feasible if the constraint (3.10) holds.*

Maximax Chance-Constrained Programming

Definition 3.4 *(Liu [44]) Let ξ be an uncertain variable, and $\alpha \in (0, 1]$. Then $\xi_{\sup}(\alpha)$ is said to be the α-optimistic value to ξ if*

$$\xi_{\sup}(\alpha) = \sup\{r | \mathcal{M}\{\xi \geq r\} \geq \alpha\},$$

and $\xi_{\inf}(\alpha)$ is said to be the α-pessimistic value to ξ if

$$\xi_{\inf}(\alpha) = \inf\{r | \mathcal{M}\{\xi \leq r\} \geq \alpha\}.$$

In order to maximize the optimistic value with a given confidence level subject to some chance constraint, the maximax chance-constrained programming (CCP) model was introduced by Liu [45] as follows,

$$\begin{cases} \max \bar{f} \\ \text{subject to:} \\ \quad \mathcal{M}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \geq \bar{f}\} \geq \alpha \\ \quad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0, j = 1, 2, \ldots, p\} \geq \beta_j \end{cases} \tag{3.11}$$

where α and β_j are the predetermined confidence levels. Model (3.11) is said to be maximax since it is equivalent to

$$\begin{cases} \max\limits_{\boldsymbol{x}} \max\limits_{\bar{f}} \bar{f} \\ \text{subject to:} \\ \quad \mathcal{M}\{f(\boldsymbol{x},\boldsymbol{\xi}) \geq \bar{f}\} \geq \alpha \\ \quad \mathcal{M}\{g_j(\boldsymbol{x},\boldsymbol{\xi}) \leq 0, j = 1,2,\ldots,p\} \geq \beta_j \end{cases} \tag{3.12}$$

where $\max\bar{f}$ is the α-optimistic value to the function $f(\boldsymbol{x},\boldsymbol{\xi})$.

Additionally, in order to address multiple objectives by means of chance constraineds, Liu [45] introduced the following maximax chance-constrained multiobjective programming (MCCMP),

$$\begin{cases} \max \left[\bar{f}_1, \bar{f}_2, \ldots, \bar{f}_m\right] \\ \text{subject to:} \\ \quad \mathcal{M}\{f_i(\boldsymbol{x},\boldsymbol{\xi}) \geq \bar{f}_i\} \geq \alpha_i, \quad i = 1,2,\ldots,m \\ \quad \mathcal{M}\{g_j(\boldsymbol{x},\boldsymbol{\xi}) \leq 0\} \geq \beta_j, \quad j = 1,2,\ldots,p \end{cases} \tag{3.13}$$

where $\alpha_i (i = 1,2,\ldots,m)$ and $\beta_j (j = 1,2,\ldots,p)$ are the predetermined confidence levels. Model (3.13) is essentially equivalent to

$$\begin{cases} \max\limits_{\boldsymbol{x}} \left[\max\limits_{\bar{f}_1} \bar{f}_1, \max\limits_{\bar{f}_2} \bar{f}_2, \ldots, \max\limits_{\bar{f}_m} \bar{f}_m\right] \\ \text{subject to:} \\ \quad \mathcal{M}\{f_i(\boldsymbol{x},\boldsymbol{\xi}) \geq \bar{f}_i\} \geq \alpha_i, & i = 1,2,\ldots,m \\ \quad \mathcal{M}\{g_j(\boldsymbol{x},\boldsymbol{\xi}) \leq 0\} \geq \beta_j, & j = 1,2,\ldots,p \end{cases} \tag{3.14}$$

where $\max\bar{f}_i$ are the α_i-optimistic values to the function $f_i(\boldsymbol{x},\boldsymbol{\xi})$, $i = 1,2,\ldots,m$, respectively.

By using goal programming, Liu [45] formulated an uncertain chance-constrained goal programming model as follows,

$$\begin{cases} \min\limits_{\boldsymbol{x}} \sum\limits_{j=1}^{l} P_j \sum\limits_{i=1}^{m} (u_{ij} d_i^+ + v_{ij} d_i^-) \\ \text{subject to:} \\ \quad \mathcal{M}\{f_i(\boldsymbol{x},\boldsymbol{\xi}) - b_i \leq d_i^+\} \geq \alpha_i^+, & i = 1,2,\ldots,m \\ \quad \mathcal{M}\{b_i - f_i(\boldsymbol{x},\boldsymbol{\xi}) \leq d_i^-\} \geq \alpha_i^-, & i = 1,2,\ldots,m \\ \quad \mathcal{M}\{g_j(\boldsymbol{x},\boldsymbol{\xi}) \leq 0\} \geq \beta_j, & j = 1,2,\ldots,p \\ \quad d_i^-, d_i^+ \geq 0, & i = 1,2,\ldots,m \end{cases} \tag{3.15}$$

where l is the number of priorities, P_j are the preemptive priority factors such that $P_j \gg P_{j+1}$ for all j, u_{ij} and v_{ij} are the weighting factors corresponding

to positive deviation and negative deviation for goal i with priority j assigned, respectively, b_i is the target value corresponding to goal i,

$$d_i^+ = \min\{d \vee 0 | \mathcal{M}\{f_i(\boldsymbol{x}, \boldsymbol{\xi}) - b_i \leq d\} \geq \alpha_i^+\}$$

is the α_i^+-optimistic positive deviation and

$$d_i^- = \min\{d \vee 0 | \mathcal{M}\{b_i - f_i(\boldsymbol{x}, \boldsymbol{\xi}) \leq d\} \geq \alpha_i^-\}$$

is the α_i^--optimistic negative deviation, f_i are functions in goal constraints, and g_j are a function in real constraints.

Minimax Chance-Constrained Programming

In order to maximize the pessimistic value with a given confidence level subject to some chance constraint, the minimax chance-constrained programming (CCP) model was introduced by Liu [45] as follows,

$$\begin{cases} \max\limits_{\boldsymbol{x}} \min\limits_{\bar{f}} \bar{f} \\ \text{subject to:} \\ \quad \mathcal{M}\{f(\boldsymbol{x}, \boldsymbol{\xi}) \leq \bar{f}\} \geq \alpha \\ \quad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0, j = 1, 2, \ldots, p\} \geq \beta_j \end{cases} \tag{3.16}$$

where α and β_j are the predetermined confidence levels, and $\min \bar{f}$ is the α-pessimistic value to the function $f(\boldsymbol{x}, \boldsymbol{\xi})$.

In order to address multiple objectives in the decision system, Liu [45] introduced the following minimax multiobjective CCP model,

$$\begin{cases} \max\limits_{\boldsymbol{x}} \left[\min\limits_{\bar{f}_1} \bar{f}_1, \min\limits_{\bar{f}_2} \bar{f}_2, \ldots, \min\limits_{\bar{f}_m} \bar{f}_m \right] \\ \text{subject to:} \\ \quad \mathcal{M}\{f_i(\boldsymbol{x}, \boldsymbol{\xi}) \leq \bar{f}_i\} \geq \alpha_i, \qquad i = 1, 2, \ldots, m \\ \quad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \geq \beta_j, \qquad j = 1, 2, \ldots, p \end{cases} \tag{3.17}$$

where α_i and β_j are the predetermined confidence levels, and $\min \bar{f}_i$ are the α_i-pessimistic values to the functions $f_i(\boldsymbol{x}, \boldsymbol{\xi})$, $i = 1, 2, \ldots, m$, respectively.

By using goal programming, Liu [45] formulated an uncertain goal programming according to the priority structure and target levels as follows,

$$\begin{cases} \min\limits_{\boldsymbol{x}} \sum\limits_{j=1}^{l} P_j \sum\limits_{i=1}^{m} \left[u_{ij} \left(\max\limits_{d_i^+} d_i^+ \vee 0 \right) + v_{ij} \left(\max\limits_{d_i^-} d_i^- \vee 0 \right) \right] \\ \text{subject to:} \\ \quad \mathcal{M}\{f_i(\boldsymbol{x}, \boldsymbol{\xi}) - b_i \geq d_i^+\} \geq \alpha_i^+, \quad i = 1, 2, \ldots, m \\ \quad \mathcal{M}\{b_i - f_i(\boldsymbol{x}, \boldsymbol{\xi}) \geq d_i^-\} \geq \alpha_i^-, \quad i = 1, 2, \ldots, m \\ \quad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \geq \beta_j, \quad j = 1, 2, \ldots, p \end{cases} \tag{3.18}$$

where l is the number of priorities, P_j are the preemptive priority factors such that $P_j \gg P_{j+1}$ for all j, u_{ij} and v_{ij} are the weighting factors corresponding to positive deviation and negative deviation for goal i with priority j assigned, respectively, b_i is the target value corresponding to goal i,

$$d_i^+ \vee 0 = \max\{d \vee 0 | \mathcal{M}\{f_i(\boldsymbol{x}, \boldsymbol{\xi}) - b_i \geq d\} \geq \alpha_i^+\}$$

is the α_i^+-pessimistic positive deviation and

$$d_i^- \vee 0 = \max\{d \vee 0 | \mathcal{M}\{b_i - f_i(\boldsymbol{x}, \boldsymbol{\xi}) \geq d\} \geq \alpha_i^-\}$$

is the α_i^--pessimistic negative deviation, f_i are functions in goal constraints, and g_j are functions in real constraints.

3.3 Measure-Chance Programming

In this section, we introduce the third type of uncertain programming, which is called measure-chance programming (MCP). A typical MCP (Liu [45]) is represented as maximizing the uncertain measure of a task subject to some chance constraints,

$$\begin{cases} \max \mathcal{M}\{h(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \\ \text{subject to:} \\ \quad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \geq \beta_j, \quad j = 1, 2, \ldots, p \end{cases} \tag{3.19}$$

where $\boldsymbol{x}$ is a decision vector, $\boldsymbol{\xi}$ is an uncertain vector, $h(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0$ represents the task, and $g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0$ are real constraints, $j = 1, 2, \ldots, p$.

If there are m tasks in a complex decision system, then a typical formulation of measure-chance multiobjective programming (MCMP) (Liu [45]) is represented as maxmizing multiple uncertain measures subject to some chance constraint,

$$\begin{cases} \max[\mathcal{M}\{h_1(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\}, \ldots, \mathcal{M}\{h_m(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\}] \\ \text{subject to:} \\ \quad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \geq \beta_j, \quad j = 1, 2, \ldots, p \end{cases} \tag{3.20}$$

where $h_i(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0$ represent m tasks, $i = 1, 2, \ldots, m$.

Measure-chance goal programming (MCGP), which can be regarded as an extension of goal programming in a complex uncertain decision system, is a frequently employed method to address MCMP. The typical formulation

(Liu [45]) can be described as follows,

$$\begin{cases} \min \sum_{j=1}^{l} P_j \sum_{i=1}^{m} \left(u_{ij} d_i^+ + v_{ij} d_i^- \right) \\ \text{subject to:} \\ \quad \mathcal{M}\{f_i(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} + d_i^- - d_i^+ = b_i, & i = 1, 2, \ldots, m \\ \quad \mathcal{M}\{g_j(\boldsymbol{x}, \boldsymbol{\xi}) \leq 0\} \geq \beta_j, & j = 1, 2, \ldots, p \\ \quad d_i^+, d_i^- \geq 0, & i = 1, 2, \ldots, m \end{cases} \tag{3.21}$$

where l is the number of priorities, P_j are the preemptive priority factors such that $P_j \gg P_{j+1}$ for all j, u_{ij} and v_{ij} are the weighting factors corresponding to positive deviation and negative deviation for goal i with priority j assigned, respectively, b_i is the target value corresponding to goal i, d_i^+ is the positive deviation and d_i^- is the negative deviation, and g_j are functions in real constraints.

Chapter 4

Uncertain Graph

As we know, the edges and vertices are all deterministic in classic graph theory. However, in practical applications, some indeterminate factors will appear not only from economic reasons, but also for technical difficulties. If the indeterminate factors come from the experts' empirical estimation, it is rational to describe the indeterminate quantities as uncertain variables.

Gao and Gao [22] first proposed the concept of uncertain graphs in which all edges are independent and exist with some belief degrees in uncertain measure. Please note that the uncertain graph is defined on a simple finite graph. For ease of understanding, we restate it in the following form.

A graph is said order of n if it has n vertices. A collection of vertices can be denoted as

$$\mathcal{V} = \{1, 2, \ldots, n\}.$$

In an uncertain environment, all deterministic edges are regarded as special uncertain ones. We will call

$$\mathcal{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}$$

an uncertain adjacency matrix if α_{ij} indicates that the edge between vertices i and j exists with uncertain measure α_{ij} and does not exist with uncertain measure $1 - \alpha_{ij}$, $i, j = 1, 2, \ldots, n$, respectively. Note that, $\alpha_{ii} = 0$ for $i = 1, 2, \ldots, n$, and the uncertain adjacency matrix is symmetric, i.e., $\alpha_{ij} = \alpha_{ji}$ for any i and j. Thus, we define a collection of uncertain edges

$$\mathcal{U} = \{(i, j) | 1 \leq i < j \leq n\}$$

and a indicator matrix

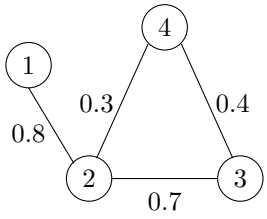
$$\xi = \begin{pmatrix} \xi_{11} & \xi_{12} & \cdots & \xi_{1n} \\ \xi_{21} & \xi_{22} & \cdots & \xi_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \xi_{n1} & \xi_{n2} & \cdots & \xi_{nn} \end{pmatrix}$$

where ξ_{ij} are uncertain Boolean variables such that $\mathcal{M}\{\xi_{ij} = 1\} = \alpha_{ij}$ and $\mathcal{M}\{\xi_{ij} = 0\} = 1 - \alpha_{ij}, i, j = 1, 2, \ldots, n$, respectively.

Definition 4.1 *Assume $\mathcal{V}$ is a collection of vertices, $\mathcal{U}$ is a collection of uncertain edges, $\mathcal{A}$ is an uncertain adjacency matrix, and ξ is an indicator matrix. Then the quartette $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ is said to be an uncertain graph.*

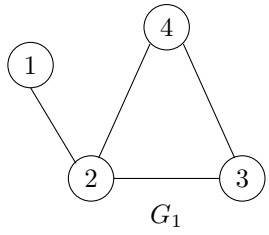
Generally, an uncertain graph $\mathcal{G}$ can be presented as Figure 4.1, where the truth values are marked on the edges. For an uncertain graph $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$, denote $\mathcal{E} = \{\xi_{ij} | 1 \leq i < j \leq n \,\text{and}\, 0 < \alpha_{ij} < 1\}$. If $|\mathcal{E}| = m$, then G contains 2^m realizations. In this section, we call each realization a model. Clearly, each model is essentially a deterministic graph $G = (V, E)$ with vertex set $V = \mathcal{V}$ and edge set $E \subset \mathcal{U}$.

For example, there are 16 models of the uncertain graph that is presented in Figure 4.1. Some models are shown in Figure 4.2.



$$\begin{pmatrix} 0 & 0.8 & 0 & 0 \\ 0.8 & 0 & 0.7 & 0.3 \\ 0 & 0.7 & 0 & 0.4 \\ 0 & 0.3 & 0.4 & 0 \end{pmatrix}$$

Figure 4.1: Uncertain graph $\mathcal{G}$ and its uncertain adjacency matrix



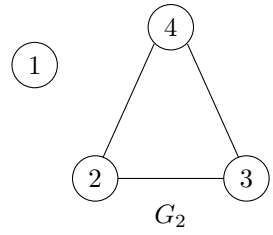


Figure 4.2: Some models of the uncertain graph $\mathcal{G}$

Definition 4.2 *(Zhang-Peng [74]) Let $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ be an uncertain graph. The underlying graph of $\mathcal{G}$, denoted by $\mathcal{G}^*$, is a graph obtained from $\mathcal{G}$ by replacing each edge in $\mathcal{E}$ with $\alpha_{ij} = 1$.*

Note that, the underlying graph is a deterministic graph that contains all the vertices and edges of the original uncertain graph. Clearly, G_1 in Figure 4.2 is just the underlying graph of the uncertain graph that is presented in Figure 4.1.

4.1 Connectivity Index

As we know, a classical graph G is said to be connected if there is a (u, v)-path in G for any two vertices u and v. In an uncertain graph, whether two vertices are joined by an edge cannot be completely determined. Further whether an uncertain graph is connected cannot be completely determined.

In order to show how likely an uncertain graph is connected, a connectedness index is defined by Gao-Gao [22] as follows.

Definition 4.3 *(Gao-Gao [22]) The connectedness index $\rho(\mathcal{G})$ of an uncertain graph $\mathcal{G}$ is the uncertain measure that $\mathcal{G}$ is connected.*

For simplicity, we denote

$$\xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$$

by removing the edges satisfying $\mathcal{M}\{\xi_{ij} = 1\} = 0$ and we call it indicator function. The edge set corresponding to ξ is denoted as

$$E = \{e_1, e_2, \ldots, e_m\}.$$

As we know, whether an uncertain graph is connected depends on the values of ξ_{ij}. Then, Gao-Gao [22] defined the following connectedness function

$$C(\xi) = \begin{cases} 1, & \text{if graph } \mathcal{G} \text{ is connected} \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $C(\xi)$ is an increasing Booleans function. According to Definition 4.3, the connectedness index of an uncertain graph $\mathcal{G}$ is

$$\rho(\mathcal{G}) = \mathcal{M}\{C(\xi) = 1\}.$$

How can we obtain the connectedness index when an uncertain graph is given? The following theorem was proposed by Gao-Gao [22] to solve this problem.

Theorem 4.1 *(Gao-Gao [22]) Let $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ be an uncertain graph, where $\xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$. The connectedness index of $\mathcal{G}$ is*

$$\rho(\mathcal{G}) = \sup_{C(\xi)=1} \min_{1 \le i \le m} \mathcal{M}\{\xi_i \in B_i\}$$

where B_i are subsets of $\{0, 1\}$, $i = 1, 2, \ldots, m$.

Proof: Obviously, since $C(\xi)$ is an increasing Boolean function, it follows from Theorem 2.8 that the theorem can be obtained immediately.

Definition 4.4 *An uncertain graph G is said to be uncertain tree if its underlying graph is a tree.*

Corollary 4.1 *(Gao-Gao [22]) Assume $\mathcal{G}$ is an uncertain tree with indicator matrix $\xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$. Then the connectedness index of $\mathcal{G}$ is the smallest value of $\mathcal{M}\{\xi_i = 1\}$, $i = 1, 2, \ldots, m$.*

Definition 4.5 *(Gao-Gao [22]) Let $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ be an uncertain graph, where $\xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$. The uncertain spanning tree of $\mathcal{G}$ is an uncertain tree T with $V(T) = \{1, 2, \ldots, n\}$ and indicator function $\xi(T) = \{\xi_{t_1}, \xi_{t_2}, \ldots, \xi_{t_{n-1}}\}$.*

Theorem 4.2 *(Gao-Gao [22]) Let $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ be an uncertain graph, where $\xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$. Then, the connectedness index of G is*

$$\rho(\mathcal{G}) = \max_{T} \rho(T) = \rho(T^*)$$

where T is an uncertain spanning tree of $\mathcal{G}$.

Proof: It is clear that $\rho(\mathcal{G}) \ge \rho(T^*)$. In the following, we will prove that $\rho(\mathcal{G}) \le \rho(T^*)$ when $\rho(\mathcal{G}) > 0$.

According to Theorem 4.1, there must exist a series of $\{\widehat{B}_i\}_{i=1}^m$, taking values of $\{1\}$ or $\{0, 1\}$, satisfying

$$\begin{aligned} \rho(\mathcal{G}) &= \sup_{C(\xi)=1} \min_{1 \le i \le m} \mathcal{M}\{\xi_i \in B_i\} \\ &= \min_{1 \le i \le m} \mathcal{M}\{\xi_i \in \widehat{B}_i\} = \min_{1 \le k \le h} \mathcal{M}\{\xi_{i_k} = 1\} > 0 \end{aligned}$$

where the subseries $\{\widehat{B}_{i_k}\}_{k=1}^h$ take values of $\{1\}$. Obviously, $h \ge n - 1$. If $h = n - 1$, then the subgraph T with vertex set $V(T) = \{1, 2, \ldots, n\}$ and edge set $E(T) = \{e_{i_1}, e_{i_2}, \ldots, e_{i_h}\}$ simply is an uncertain spanning tree, and we get

$$\rho(\mathcal{G}) = \min_{1 \le k \le h} \mathcal{M}\{\xi_{i_k} = 1\} = \rho(T).$$

Since T^* is an uncertain spanning tree with maximum connectedness index, then

$$\rho(\mathcal{G}) = \rho(T) \le \rho(T^*).$$

If $h > n-1$, let S be a subgraph with vertex set $V(S) = \{1, 2, \ldots, n\}$ and edge set $E(S) = \{e_{i_1}, e_{i_2}, \ldots, e_{i_h}\}$. Then it must contain at least an uncertain spanning tree and an uncertain circle C. Choose any edge of C such as e_{i_h} and then remove it. A new subgraph S_1 with vertex set $V(S_1) = \{1, 2, \ldots, n\}$ and edge set $E(S_1) = \{e_{i_1}, e_{i_2}, \ldots, e_{i_{h-1}}\}$ is obtained. Obviously, S_1 must also contain an uncertain spanning tree and

$$\rho(\mathcal{G}) \geq \rho(S_1) \geq \min_{1 \leq k \leq h-1} \mathcal{M}\{\xi_{i_k} = 1\} \geq \min_{1 \leq k \leq h} \mathcal{M}\{\xi_{i_k} = 1\} = \rho(\mathcal{G}).$$

That is,

$$\rho(\mathcal{G}) = \rho(S_1).$$

Repeating this argument until there is no circle. At last, we can obtain an spanning tree T' such that

$$\rho(\mathcal{G}) = \rho(T').$$

Since T^* is an uncertain spanning tree with maximum connectedness index, then

$$\rho(\mathcal{G}) = \rho(T') \leq \rho(T^*).$$

The theorem is proved.

Based on Theorem 4.2, Gao-Gao [22] proposed two algorithms to obtain the connectedness index for an uncertain graph.

The first algorithm comes from Prim's Algorithm. Assume uncertain graph $\mathcal{G}$ has vertex set $V(\mathcal{G}) = \{1, 2, \ldots, n\}$, indicator function $\xi(\mathcal{G}) = \{\xi_1, \xi_2, \ldots, \xi_m\}$, and edge set $E(\mathcal{G}) = \{e_1, e_2, \ldots, e_m\}$.

Algorithm 4.1 (Connectedness Index Algorithm-1)

Step 1. Set $V = \{1\}$, $U = V(\mathcal{G}) - V$ and $E = \emptyset$.

Step 2. For all edges $e_i = (s_i, t_i)$ satisfying $s_i \in V$ and $t_i \in U$, choose the edge with biggest $\mathcal{M}\{\xi_i = 1\}$, such as $e_1 = (s_1, t_1)$. Reset $V = V \cup \{t_1\}$, $U = U - \{t_1\}$ and $E = E \cup \{e_1\}$.

Step 3. Repeat Step 2 until $V = V(\mathcal{G})$.

After the last iteration, an uncertain spanning tree T with vertex set $V(T) = V(\mathcal{G})$ is obtained. It is easy to verify that T an uncertain spanning tree with maximum connectedness index. That is, $\rho(\mathcal{G}) = \rho(T)$.

In addition, Algorithm 4.1 has the same complexity as Prim's Algorithm, that is, $O(n^2)$. Figure 4.3 shows an example to obtain an uncertain spanning tree with maximum connectedness index by Algorithm 4.1. According to Corollary 4.1, the connectedness index of the uncertain graph in Figure 4.3(1) is 0.7.

In addition, the second algorithm comes from Kruskal's Algorithm. Assume T is a subgraph of the underlying graph of $\mathcal{G}$.

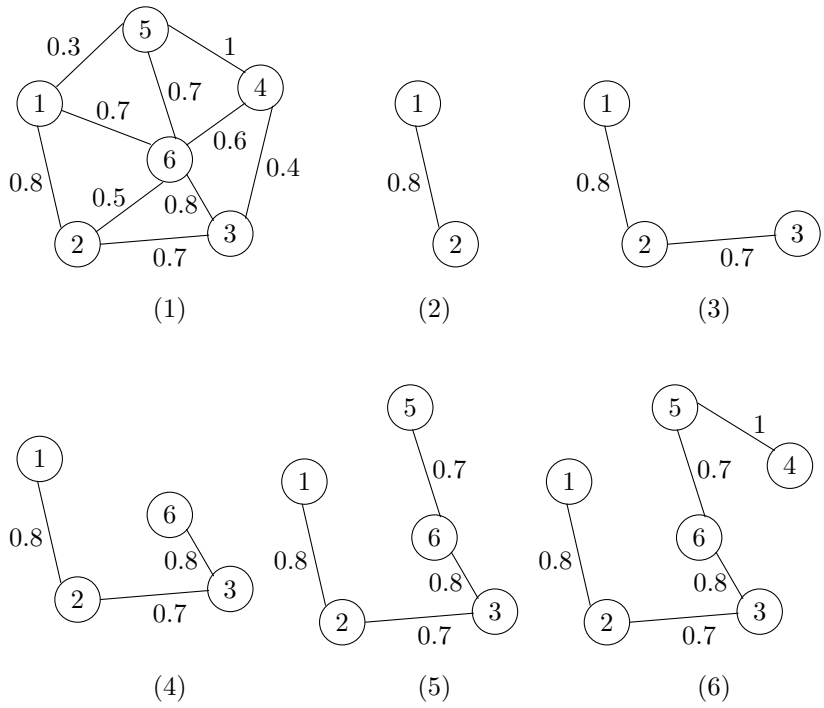


Figure 4.3: An example for Algorithm 4.1

Algorithm 4.2 (Connectedness Index Algorithm-2)

Step 1. Set $V(T) = V(\mathcal{G})$, $E(T) = \emptyset$ and $E' = \{e_1, e_2, \ldots, e_m\}$.

Step 2. Choose the edge e_i with biggest $\mathcal{M}\{\xi_i = 1\}$ in E'. If e_i joins two components of T, reset $E(T) = E(T) \cup \{e_i\}$, $E' = E' - \{e_i\}$; otherwise, reset $E' = E' - \{e_i\}$.

Step 3. Repeat Step 2 until $|E(T)| = n - 1$.

Similarly, the subgraph T obtained by Algorithm 4.2 is simply an uncertain spanning tree with maximum connectedness index. Clearly, Algorithm 4.2 has the same complexity as Kruskal's Algorithm, that is, $O(m \log m)$. Figure 4.4 shows an example to obtain an uncertain spanning tree with maximum connectedness index by Algorithm 4.2. According to Corollary 4.1, the connectedness index of the uncertain graph in Figure 4.4(1) is 0.7.

Sometimes, someone may be interested in whether or not two specific vertices are connected. In order to show the connectivity index of two vertices, we will introduce some basic concepts of the uncertain graph.

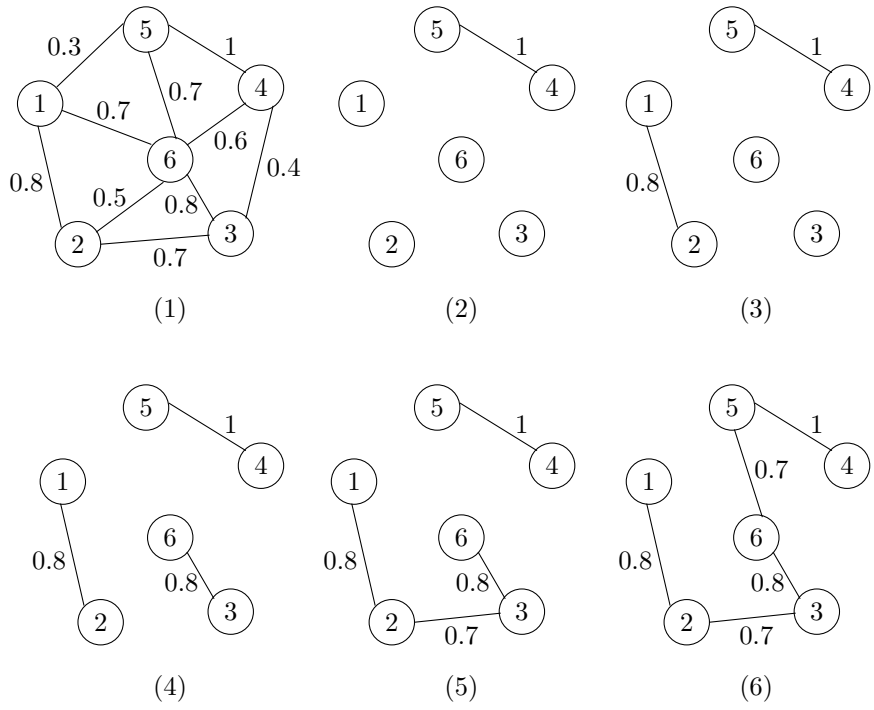


Figure 4.4: An example for Algorithm 4.2

Definition 4.6 *(Zhang-Peng [73]) An uncertain graph $\mathcal{G}$ is said to be an uncertain path if its underlying graph is a path.*

Definition 4.7 *(Zhang-Peng [73]) The strength of an uncertain path is the uncertain measure that the uncertain path exists.*

Theorem 4.3 *(Zhang-Peng [73]) Assume P is an uncertain path with indicator function $\xi = (\xi_1, \xi_2, \ldots, \xi_m)$, then the strength of P is the smallest value of $\mathcal{M}\{\xi_i = 1\}$, i.e.,*

$$Str(P) = \bigwedge_{i=1}^{m} \mathcal{M}\{\xi_i = 1\}.$$

Proof: The proof follows from the fact that the uncertain path P exists if and only if all uncertain variables ξ_i equal to 1.

In order to measure how likely two vertices of an uncertain graph are connected, a connectedness strength of two vertices is defined as follows.

Definition 4.8 *(Zhang-Peng [73]) The connectedness strength of two vertices i and j in an uncertain graph is the uncertain measure that i and j are connected.*

Theorem 4.4 *(Zhang-Peng [73]) Let $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ be an uncertain graph. If all edges are independent, then the connectedness strength of two vertices i and j of $\mathcal{G}$ is*

$$\rho_{\mathcal{G}}(i,j) = \begin{cases} \sup\limits_{f(X)=1} \min\limits_{1\le i<j\le n} \nu_{ij}(X), & \text{if } \sup\limits_{f(X)=1} \min\limits_{1\le i<j\le n} \nu_{ij}(X) < 0.5 \\ 1 - \sup\limits_{f(X)=0} \min\limits_{1\le i<j\le n} \nu_{ij}(X), & \text{if } \sup\limits_{f(X)=1} \min\limits_{1\le i<j\le n} \nu_{ij}(X) \ge 0.5 \end{cases}$$

where

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix},$$

$x_{ij} \in \{0,1\}$, $x_{ij} = x_{ji}$, $x_{ii} = 0$, and ν_{ij} are defined by

$$\nu_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases}$$

for $i, j = 1, 2, \ldots, n$, respectively, and

$$f(X) = \begin{cases} 1, & \text{if there exists a nonnegative integer } t \text{ such that } (X^t)_{ij} > 0 \\ 0, & \text{otherwise.} \end{cases}$$

Proof: All edges are essentially independent Boolean uncertain variables, and are represented by

$$\xi_{ij} = \begin{cases} 1, & \text{with uncertain measure } \alpha_{ij} \\ 0, & \text{with uncertain measure } 1 - \alpha_{ij} \end{cases}$$

for $i, j = 1, 2, \ldots, n$. As we know, two vertices i and j are connected if and only if there exists a nonnegative integer t such that $(\xi^t)_{ij} > 0$. Thus, the connectedness strength of vertices i and j is

$$\rho_{\mathcal{G}}(i,j) = \mathcal{M}\{f(\xi) = 1\}.$$

Since the function f is Boolean, it follows from Theorem 2.5 that the theorem is proved.

Theorem 4.5 *(Zhang-Peng [73]) Let $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ be an uncertain graph, and $P_1, P_2, \ldots, P_k$ be uncertain paths from i to j in $\mathcal{G}$. Then the connectedness strength of i and j is the largest strength of P_m, i.e.,*

$$\rho_{\mathcal{G}}(i,j) = \bigvee_{m=1}^{k} Str(P_m),$$

$i, j = 1, 2, \ldots, n$, respectively.

Proof: The theorem follows from the fact that if one uncertain path P_m is contained within the graph, then the vertices i and j in $\mathcal{G}$ are connected.

Example 4.1: Assume an uncertain graph of order 3 and size 3 and its uncertain adjacency matrix is

$$\mathcal{A} = \begin{pmatrix} 0 & 0.7 & 0.6 \\ 0.7 & 0 & 0.8 \\ 0.6 & 0.8 & 0 \end{pmatrix}.$$

Then the strength of the uncertain path $P_1 : 1 \to 2$ is 0.7, the strength of the uncertain path $P_2 : 1 \to 3 \to 2$ is 0.6. Thus, we have the connectedness strength of 1 and 2 is 0.7.

Example 4.2: Assume an uncertain graph $\mathcal{G}$ has an uncertain adjacency matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 0.4 & 0.8 & 0.9 & 0.5 & 0.6 \\ 0.4 & 0 & 0.6 & 0.5 & 0.2 & 0.7 \\ 0.8 & 0.6 & 0 & 0.8 & 0.3 & 0.4 \\ 0.9 & 0.5 & 0.8 & 0 & 0.9 & 0.8 \\ 0.5 & 0.2 & 0.3 & 0.9 & 0 & 0.3 \\ 0.6 & 0.7 & 0.4 & 0.8 & 0.3 & 0 \end{pmatrix}.$$

According to Theorem 4.5, the connectedness strength of 1 and 3 is $\rho_{\mathcal{G}}(1,3) = 0.8$, which is obtained by MATLAB.

The following theorem shows the relationship between the connectedness strength of two vertices and the connectedness index of the uncertain graph.

Theorem 4.6 *(Zhang-Peng [73]) Assume $\mathcal{G}$ is an uncertain graph with vertex set $V = \{1, 2, \ldots, n\}$. Then the connectedness index of $\mathcal{G}$ is the smallest connectedness strength of two vertices in $\mathcal{G}$, i.e.,*

$$\rho(\mathcal{G}) = \min_{1 \le i < j \le n} \rho_G(i, j),$$

$i, j = 1, 2, \ldots, n$, respectively.

Proof: The theorem follows from the fact that $\mathcal{G}$ is connected if and only if each pair of vertices of $\mathcal{G}$ are connected.

Theorem 4.6 provides a new method to obtain connectivity index of uncertain graph $\mathcal{G}$.

Example 4.3: Assume an uncertain graph $\mathcal{G}$ has an uncertain adjacency matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 0.6 & 0.2 & 0.8 \\ 0.6 & 0 & 0.9 & 0.4 \\ 0.2 & 0.9 & 0 & 0.6 \\ 0.8 & 0.4 & 0.6 & 0 \end{pmatrix}.$$

The connectedness strength of each pair of vertices are shown in Table 4.1. According to Theorem 4.6, the connectivity index of the uncertain graph $\mathcal{G}$ is

$$\rho(\mathcal{G}) = \min_{1\leq i<j\leq 4} \rho_{\mathcal{G}}(i,j) = 0.6.$$

Table 4.1: List of the connectedness strength of two vertices i and j

Pair of vertices (i,j)	$\rho_{\mathcal{G}}(i,j)$
(1,2)	0.6
(1,3)	0.6
(1,4)	0.8
(2,3)	0.9
(2,4)	0.6
(3,4)	0.6

4.2 Euler Index

In classical graph theory, a closed walk that traverses each edge of a graph G at least once is called a tour of G. An Euler tour is a tour which traverses each edge exactly once. A graph is Eulerian if it contains an Euler tour, and non-Eulerian otherwise. It is easy to verify that a nonempty connected graph is Eulerian if and only if it has no vertices of odd degree. The degree of a vertex v denotes the number of edges of G that incident with v.

In order to show how likely an uncertain graph is Eulerian, Zhang-Peng [71] proposed an Euler index as follows.

Definition 4.9 *(Zhang-Peng [71]) The Euler index of an uncertain graph is the uncertain measure that the uncertain graph is Eulerian.*

If a graph is Eulerian, then it must be connected. But the converse is not necessarily true. Thus, we have the following result.

Remark 4.1: Let $\mathcal{G}$ be an uncertain graph, then the Euler index of $\mathcal{G}$ is less than or equal to the connectedness index of $\mathcal{G}$.

The following theorem calculates the Euler index.

Theorem 4.7 *(Zhang-Peng [71]) Let $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ be an uncertain graph. If all edges are independent, then the Euler index of $\mathcal{G}$ is*

$$\mu(\mathcal{G}) = \begin{cases} \sup\limits_{g(X)=1} \min\limits_{1\leq i<j\leq n} \nu_{ij}(X), & \text{if } \sup\limits_{g(X)=1} \min\limits_{1\leq i<j\leq n} \nu_{ij}(X) < 0.5 \\ 1 - \sup\limits_{g(X)=0} \min\limits_{1\leq i<j\leq n} \nu_{ij}(X), & \text{if } \sup\limits_{g(X)=1} \min\limits_{1\leq i<j\leq n} \nu_{ij}(X) \geq 0.5 \end{cases}$$

where

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix},$$

$x_{ij} \in \{0, 1\}$, $x_{ij} = x_{ji}$, $x_{ii} = 0$, *and* ν_{ij} *are defined by*

$$\nu_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases}$$

for $i, j = 1, 2, \ldots, n$, *respectively, and*

$$g(X) = \begin{cases} 1, & \text{if } \begin{cases} I + X + X^2 + \cdots + X^{n-1} > 0 \\ \sum\limits_{1 \le j \le n} x_{ij} \text{ are even for } i = 1, 2, \ldots, n \end{cases} \\ 0, & \text{otherwise.} \end{cases}$$

Proof: It is clear that the uncertain graph is connected if and only if

$$I + \xi + \xi^2 + \cdots + \xi^{n-1} > 0.$$

Since the degree of a vertex i is just the sum of entries in the row corresponding to vertex i in ξ, then the uncertain graph is Eulerian if and only if

$$g(\xi) = 1.$$

It follows from the definition of the Euler index that

$$\mu(\mathcal{G}) = \mathcal{M}\{g(\xi) = 1\}.$$

Note that all edges are essentially independent Boolean uncertain variables, and g is Boolean, it follows from Theorem 2.5 that the theorem is verified.

Theorem 4.7 provides a basic method to obtain the Euler index for an uncertain graph. Next, we will introduce a new method to calculate the Euler index.

Theorem 4.8 *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ *be an uncertain graph. If all edges are independent, then the Euler index of* $\mathcal{G}$ *is*

$$\mu(\mathcal{G}) = \sup_{g(X)=1} \min_{1 \le i < j \le n} \left\{ \min_{x_{ij}=1} \alpha_{ij}, 1 - \max_{x_{ij}=0} \alpha_{ij} \right\}.$$

Proof: According to Theorem 4.7, if

$$\sup_{g(X)=1} \min_{1 \le i < j \le n} \nu_{ij}(X) < 0.5,$$

it is easy to verify that

$$\mu(\mathcal{G}) = \sup_{g(X)=1} \min_{1\le i<j\le n} \left\{ \min_{x_{ij}=1} \alpha_{ij}, 1 - \max_{x_{ij}=0} \alpha_{ij} \right\}.$$

If

$$\sup_{g(X)=1} \min_{1\le i<j\le n} \nu_{ij}(X) \ge 0.5,$$

there must exist a matrix X' such that $g(X') = 1$ and

$$\min_{1\le i<j\le n} \nu_{ij}(X') = \sup_{g(X)=1} \min_{1\le i<j\le n} \nu_{ij}(X).$$

Denote G' as the graph that corresponding to the adjacency matrix X', and $\mathcal{G}^*$ as the underlying graph of $\mathcal{G}$. Suppose that G' contains k edges, and $\mathcal{G}^*$ contains m edges. Obviously, $k \le m$. Now, we rearrange the edges of $\mathcal{G}$ such that the edge $e_i \in E(G')$, $i = 1, 2, \ldots, k$, and

$$\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_k \ge 0.5 \ge \alpha_{k+1} \ge \cdots \ge \alpha_m$$

where α_i show the truth values in uncertain measure that edges e_i exist, $i = 1, 2, \ldots, m$. Then, we have

$$\begin{aligned} \sup_{g(X)=0} \min_{1\le i<j\le n} \nu_{ij}(X) &= \sup_{g(X)=0} \min_{1\le i<j\le n} \left\{ \min_{x_{ij}=1} \alpha_{ij}, 1 - \max_{x_{ij}=0} \alpha_{ij} \right\} \\ &= \max\left\{1 - \alpha_k, \alpha_{k+1}\right\}. \end{aligned}$$

According to Theorem 4.7, if $1 - \alpha_k \ge \alpha_{k+1}$, we have

$$\mu(\mathcal{G}) = 1 - (1 - \alpha_k) = \alpha_k = \sup_{g(X)=1} \min_{1\le i<j\le n} \left\{ \min_{x_{ij}=1} \alpha_{ij}, 1 - \max_{x_{ij}=0} \alpha_{ij} \right\}.$$

If $1 - \alpha_k < \alpha_{k+1}$, we have

$$\mu(\mathcal{G}) = 1 - \alpha_{k+1} = \sup_{g(X)=1} \min_{1\le i<j\le n} \left\{ \min_{x_{ij}=1} \alpha_{ij}, 1 - \max_{x_{ij}=0} \alpha_{ij} \right\}.$$

Thus, we prove the theorem.

Example 4.4: Let $\mathcal{G}$ be an uncertain graph of order 5 and size 10 has an uncertain adjacency matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 0.3 & 0.6 & 0.7 & 0.2 \\ 0.3 & 0 & 0.8 & 0.3 & 0.6 \\ 0.6 & 0.8 & 0 & 0.3 & 0.7 \\ 0.7 & 0.3 & 0.3 & 0 & 0.4 \\ 0.2 & 0.6 & 0.7 & 0.4 & 0 \end{pmatrix}.$$

Employing the method mentioned above, we get the Euler index $\mu(\mathcal{G}) = 0.3$.

In addition, Zhang-Peng [71] discussed the Euler index of some special uncertain graphs.

Theorem 4.9 *(Zhang-Peng [71]) Let $\mathcal{G}$ be an uncertain graph whose underlying graph $\mathcal{G}^*$ is a circle, denote $\mathcal{A} = (\alpha_{ij})_{n\times n}$ as the uncertain adjacency matrix. Then the Euler index of $\mathcal{G}$ is the smallest positive value of α_{ij}, for $i, j = 1, 2, \ldots, n$, respectively.*

Proof: Since $\mathcal{G}$ is an uncertain graph whose underlying graph $\mathcal{G}^*$ is a circle, then $\mathcal{G}$ is Eulerian if and only if it contains all the edges of the original uncertain graph. Hence the theorem follows from Definition of 4.9 immediately.

Example 4.5: Assume an uncertain graph G has an uncertain adjacency matrix

$$\mathcal{A} = \begin{pmatrix} 0 & 0.3 & 0.7 \\ 0.3 & 0 & 0.8 \\ 0.7 & 0.8 & 0 \end{pmatrix}.$$

It follows from Theorem 4.9 that the Euler index of G is $\mu(G) = 0.3$.

4.3 Matching Index

In this section, we will discuss the matching in uncertain graphs.

Definition 4.10 *(Zhang-Peng [74]) Let $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ be an uncertain graph with underlying graph $\mathcal{G}^*$, and M a maximum matching of $\mathcal{G}^*$. The matching function of $\mathcal{G}$ is denoted as:*

$$P(\xi) = \begin{cases} 1, & \text{if there is a matching } M' \text{ in } \mathcal{G} \text{ such that } |M'| = |M| \\ 0, & \text{otherwise.} \end{cases}$$

Definition 4.11 *(Zhang-Peng [74]) Let $\mathcal{G}$ be an uncertain graph with underlying graph $\mathcal{G}^*$, and M be a maximum matching of $\mathcal{G}^*$. The matching index $\kappa(\mathcal{G})$ of the uncertain graph $\mathcal{G}$ is the uncertain measure that $\mathcal{G}$ has a matching M' such that $|M'| = |M|$, i.e.,*

$$\kappa(\mathcal{G}) = \mathcal{M}\{P(\xi) = 1\}.$$

Theorem 4.10 *(Zhang-Peng [74]) Assume $\mathcal{G}$ is an uncertain graph of order n and its uncertain adjacency matrix is*

$$\mathcal{A} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{pmatrix}$$

If all edges are independent, then the matching index of $\mathcal{G}$ is

$$\kappa(G) = \begin{cases} \sup\limits_{P(X)=1} \min\limits_{1\le i<j\le n} \nu_{ij}(X), & \text{if } \sup\limits_{P(X)=1} \min\limits_{1\le i<j\le n} \nu_{ij}(X) < 0.5 \\ 1 - \sup\limits_{P(X)=0} \min\limits_{1\le i<j\le n} \nu_{ij}(X), & \text{if } \sup\limits_{P(X)=1} \min\limits_{1\le i<j\le n} \nu_{ij}(X) \ge 0.5 \end{cases}$$

where

$$X = \begin{pmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \cdots & x_{nn} \end{pmatrix},$$

$x_{ij} \in \{0,1\}$, $x_{ij} = x_{ji}$, $x_{ii} = 0$, and ν_{ij} are defined by

$$\nu_{ij}(X) = \begin{cases} \alpha_{ij}, & \text{if } x_{ij} = 1 \\ 1 - \alpha_{ij}, & \text{if } x_{ij} = 0 \end{cases}$$

for $i, j = 1, 2, \ldots, n$, respectively, and

$$P(X) = \begin{cases} 1, & \text{if there is a matching } M' \text{ in } X \text{ such that } |M'| = |M| \\ 0, & \text{otherwise} \end{cases}$$

where M is the maximum matching of the underlying graph $\mathcal{G}^*$.

Proof: It is clear that $\mathcal{G}$ has a matching M' such that $|M'| = |M|$ if and only if $P(X) = 1$. Note that $P(X)$ is a Boolean function. It follows from Theorem 2.5 that the theorem is proved.

According to the definition of matching index, Theorem 2.8 directly leads to

Theorem 4.11 *(Zhang-Peng [74]) Let $\mathcal{G}$ be an uncertain graph with indicator function $\xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$. The matching index of $\mathcal{G}$ is*

$$\kappa(\mathcal{G}) = \sup_{P(\xi)=1} \min_{1\le i\le m} \mathcal{M}\{\xi_i \in B_i\}$$

where B_i are subsets of $\{0,1\}$, $i = 1, 2, \ldots, m$.

Proof: The result immediately follows from Theorem 2.8, since $P(\xi)$ is an increasing Boolean function.

Corollary 4.2 *(Zhang-Peng [74]) Let $\mathcal{G}$ be an uncertain graph with indicator function $\xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$. If no two edges are adjacent in $\mathcal{G}$. Then the matching index of $\mathcal{G}$ is the smallest value of $\mathcal{M}\{\xi_i = 1\}$, $i = 1, 2, \ldots, m$.*

Definition 4.12 *(Zhang-Peng [74]) Assume that $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ is an uncertain graph with underlying graph $\mathcal{G}^*$, and M is a maximum matching of $\mathcal{G}^*$. A subset M' of uncertain edge set $\mathcal{U}(\mathcal{G})$ is called a spanning matching of $\mathcal{G}$ if no two of elements of M' are adjacent in $\mathcal{G}$, and $|M'| = |M|$. The maximum index spanning matching is a spanning matching with maximum matching index.*

According to the definition of the maximum index spanning matching, another method to calculate the matching index for other more complex uncertain graphs is given as follows.

Theorem 4.12 *(Zhang-Peng [74]) Let $\mathcal{G}$ be an uncertain graph with indicator function $\xi(\mathcal{G}) = \{\xi_1, \xi_2, \ldots, \xi_m\}$, edge set $E(\mathcal{G}) = \{e_1, e_2, \ldots, e_m\}$, and M^* be the maximum index spanning matching of $\mathcal{G}$. Then we have*

$$\kappa(\mathcal{G}) = \kappa(M^*).$$

Proof: Obviously, $\kappa(\mathcal{G}) \geq \kappa(M^*)$. We only need to prove $\kappa(\mathcal{G}) \leq \kappa(M^*)$ where $\kappa(\mathcal{G}) > 0$.

Since $P(\xi(\mathcal{G}))$ is an increasing Boolean function, according to Theorem 4.11, there must exist a series of $\{B_i'\}_{i=1}^m$, taking values of $\{1\}$ or $\{0, 1\}$, satisfying

$$\begin{aligned} \kappa(\mathcal{G}) &= \sup_{P(\xi(\mathcal{G}))=1} \min_{1\leq i\leq m} \mathcal{M}\{\xi_i \in B_i\} \\ &= \min_{1\leq i\leq m} \mathcal{M}\{\xi_i \in B_i'\} = \min_{1\leq k\leq t} \mathcal{M}\{\xi_{i_k} = 1\} > 0 \end{aligned} \tag{4.1}$$

where t is a positive number and subseries $\{B_{i_k}'\}_{k=1}^t$ takes values of $\{1\}$.

Obviously, $t \geq |M^*|$. In fact, we can choose $\{B_{i_k}'\}_{k=1}^t$ such that $t = |M^*|$.

Assume $t > |M^*|$, and S is a subgraph with edge set $E(S) = \{e_{i_1}, e_{i_2}, \ldots, e_{i_t}\}$. Then it must contain a spanning matching and some other edges, which are denoted by a set A. Choose any edge of A, such as e_{i_t}, and then remove it. A new subgraph S_1 with edge set $E(S_1) = \{e_{i_1}, e_{i_2}, \ldots, e_{i_{t-1}}\}$ can be obtained. Clearly, S_1 must contain a spanning matching and

$$\kappa(\mathcal{G}) \geq \kappa(S_1) \geq \min_{1\leq k\leq t-1} \mathcal{M}\{\xi_{i_k} = 1\} \geq \min_{1\leq k\leq t} \mathcal{M}\{\xi_{i_k} = 1\}. \tag{4.2}$$

According to the equations (4.1) and (4.2), we have

$$\kappa(\mathcal{G}) = \kappa(S_1).$$

Repeating this argument until there is only a spanning matching, we obtain a spanning matching M such that

$$\kappa(\mathcal{G}) = \kappa(M).$$

Since M^* is the maximum index spanning matching, we have

$$\kappa(\mathcal{G}) = \kappa(M) \leq \kappa(M^*).$$

This completes the proof.

Remark 4.2: Theorem 4.12 tells us that we should find the maximum index spanning matching if we want to calculate the matching index.

Assume that uncertain graph $\mathcal{G}$ has vertex set $\mathcal{V}(\mathcal{G}) = \{1, 2, \ldots, n\}$, edge set $E(\mathcal{G}) = \{e_1, e_2, \ldots, e_m\}$, and indicator function $\xi(\mathcal{G}) = \{\xi_1, \xi_2, \ldots, \xi_m\}$. Suppose that M is a maximum matching of the underlying graph $\mathcal{G}^*$. A matching index algorithm was designed by Zhang-Peng [74] to search a maximum index spanning matching of $\mathcal{G}$.

Algorithm 4.3 (Matching Index Algorithm)

Step 1. Sort the edges $\{e_1, e_2, \ldots, e_m\}$ by $\mathcal{M}\{\xi_i = 1\}$. Set $E = \emptyset$, $V = \emptyset$ and $E' = \{e_1, e_2, \ldots, e_m\}$.

Step 2. Choose the edges with the biggest $\mathcal{M}\{\xi_i = 1\}$ in E', and denote the set of these selected edges as A. The vertex set corresponding to A is denoted as K. Constructing a graph G' with edge set $E = E \cup A$ and vertex set $V = V \cup K$, and resetting $E' = E' \setminus A$.

Step 3. Find a maximum matching M' in the graph G'. The matching M' is the maximum index spanning matching if $|M'| = |M|$; otherwise, go back to Step 2.

In the algorithm 4.3, many effective algorithms, such as Edmonds's matching algorithm (Edmonds [15]), can be used to find a maximum matching in Step 3.

Theorem 4.13 *(Zhang-Peng [74]) For a given uncertain graph $\mathcal{G}$, Algorithm 4.3 creates a maximum index spanning matching.*

Proof: In Algorithm 4.3, Step 3 creates a spanning matching M' such that $|M'| = |M|$. Let M^* be a maximum index spanning matching. If $M' = M^*$, then a maximum index spanning matching is obtained.

Otherwise, there must exist a walk $(v_0 e_0 v_1 e_1 v_2 \ldots e_{2t} v_{2t+1} e_{2t+1})$ in $\mathcal{G}$ such that $E_0 = \{e_0, e_2, \ldots, e_{2t}\} \in M'$, $E_1 = \{e_1, e_3, \ldots, e_{2t+1}\} \in M^*$, and $M_1^* = M^* + E_0 - E_1$ is also a matching. In addition, Algorithm 4.3 tells that

$$\min_{0 \le k \le t} \mathcal{M}\{\xi_{2k} = 1\} \ge \min_{0 \le k \le t} \mathcal{M}\{\xi_{2k+1} = 1\}.$$

It follows from Corollary 4.2 that M_1^* is a spanning matching with

$$\kappa(M_1^*) \ge \kappa(M^*).$$

Since M^* is the maximum index spanning matching,

$$\kappa(M_1^*) = \kappa(M^*).$$

That is, we obtain a maximum index spanning matching M_{I}^* that agrees more with M' than M^* does.

Repeating this argument, we will finally obtain a maximum index spanning matching that agrees completely with M'. That is, the spanning matching obtained by Algorithm 4.3 is the maximum index spanning matching.

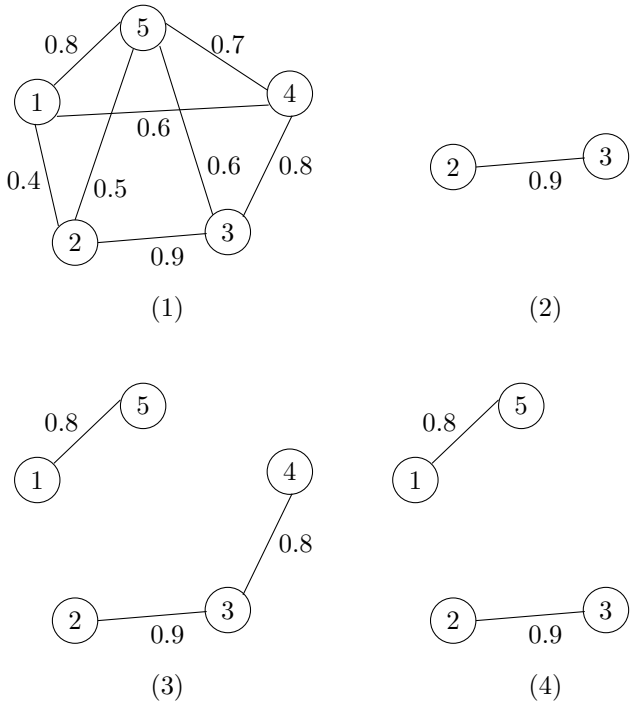


Figure 4.5: Maximum index spanning matching by Algorithm 4.3

Figure 4.5 gives an example which achieves the maximum index spanning matching by Algorithm 4.3. At the last iteration, Algorithm 4.3 creates a maximum matching which is shown in Figure 4.5(4). It following from Corollary 4.2, the matching index of the maximum index spanning matching is 0.8, thus the matching index of the uncertain graph in Figure 4.5(1) is 0.8.

4.4 Matching Number

The number of edges of a maximum matching in a graph is called the matching number of the graph. Usually, we denote by $\alpha(G)$ the matching number of a graph G. Obviously, the matching number is a constant for a classic graph. However, it is not the case for an uncertain graph. For an uncertain graph, different models may lead to different matching number. For example,

in Figure 4.2, $\alpha(G_1) = 2$ and $\alpha(G_2) = 1$. That is, for an uncertain graph $\mathcal{G}$, matching number $\alpha(\mathcal{G})$ is an uncertain variable rather than a crisp value.

Definition 4.13 *(Li-Zhang-Peng [41]) An uncertain graph $\mathcal{G}$ is said to be k-edge matching if $\alpha(\mathcal{G}) \geq k$ for a given positive integer k.*

In order to indicate the belief degree that an uncertain graph $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ is k-edge matching, it is meaningful to calculate the value of

$$\mathcal{M}\{\alpha(\mathcal{G}) \geq k\}.$$

Obviously, $\alpha(\mathcal{G})$ can be considered as a function of ξ. Thus, we can denote $\alpha(\mathcal{G}) = \alpha(\xi_1, \xi_2, \ldots, \xi_m)$. It is easy to verify that $\alpha(\xi_1, \xi_2, \ldots, \xi_m)$ is a strictly increasing function of ξ_i, $i = 1, 2, \ldots, m$. According to the definition of k-edge matching, Theorem 2.7 directly leads to

Theorem 4.14 *(Li-Zhang-Peng [41]) Let $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ be an uncertain graph. For any positive integer k, we have*

$$\mathcal{M}\{\alpha(\xi_1, \xi_2, \ldots, \xi_m) \geq k\} = \sup_{\alpha(\mathcal{G}) \subset [k, +\infty)} \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\},$$

where B_i is the subset of $\{0, 1\}$, $i = 1, 2, \ldots, m$, respectively.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{U}, \mathcal{A}, \xi)$ be an uncertain graph and we define

$$\mathcal{N}(k) = \{H | H \text{ is a model of } \mathcal{G} \text{ and } \alpha(H) \geq k\}.$$

By integrating the set $\mathcal{N}(k)$, Li-Zhang-Peng [41] provide the following alternative method to calculate the uncertain measure that an uncertain graph is k-edge matching.

Theorem 4.15 *(Li-Zhang-Peng [41]) Let $\mathcal{G}$ be an uncertain graph with edge set $E(\mathcal{G}) = \{e_1, e_2, \ldots, e_m\}$ and indicator function $\xi = \{\xi_1, \xi_2, \ldots, \xi_m\}$. The following equation holds*

$$\mathcal{M}\{\alpha(\mathcal{G}) \geq k\} = \sup_{H \in \mathcal{N}(k)} \min_{e_i \in E(H)} \mathcal{M}\{\xi_i = 1\} = \sup_{H \in \mathcal{N}(k)} \min_{e_i \in E(H)} \alpha_i$$

for any given positive integer k.

Proof: According to Theorem 4.14, the proof can be divided into the following two steps.

Step 1: We will prove the following inequality holds

$$\sup_{H \in \mathcal{N}(k)} \min_{e_i \in E(H)} \mathcal{M}\{\xi_i = 1\} \geq \sup_{\alpha(\mathcal{G}) \subset [k, +\infty)} \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B\}. \tag{4.3}$$

Since m is a finite integer, there exists a series $\{B_1^*, B_2^*, \ldots, B_m^*\}$, where $B_i^* \in \{\{0\}, \{1\}, \{0,1\}\}, i = 1, 2, \ldots, m$, such that

$$\alpha(B_1^*, \ldots, B_m^*) \subset [k, \frac{n}{2}],$$

and

$$\sup_{\alpha(B_1, B_2, \ldots, B_m) \subset [k, \frac{n}{2}]} \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i^*\}.$$

Without loss of generality, suppose $B_1^* = \{0\}$, we have,

$$\alpha(\{0\}, B_2^*, \ldots, B_m^*) \subset [k, \frac{n}{2}].$$

Thus we have the fact that

$$\alpha(\{0,1\}, B_2^*, \ldots, B_m^*) \subset [k, \frac{n}{2}],$$

since $\alpha(\xi_1, \xi_2, \ldots, \xi_m)$ is strictly increasing with respect to ξ_i. On the one hand,

$$\min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i^*\} \leq \mathcal{M}\{\xi_1 \in \{0,1\}\} \wedge \min_{2 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i^*\}.$$

On the other hand,

$$\sup_{\alpha(B_1, B_2, \ldots, B_m) \subset [k, \frac{n}{2}]} \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B\} \geq \mathcal{M}\{\xi_1 \in \{0,1\}\} \wedge \min_{2 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i^*\}.$$

Thus, it is easy to verify that

$$\begin{aligned} \sup_{\alpha(B_1, B_2, \ldots, B_m) \subset [k, \frac{n}{2}]} \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\} &= \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i^*\} \\ &= \mathcal{M}\{\xi_1 \in \{0,1\}\} \wedge \min_{2 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i^*\}. \end{aligned}$$

That is, by extending B_i^* from $\{0\}$ to $\{0,1\}$, we can choose $\{B_1^*, B_2^*, \ldots, B_m^*\}$, where $B_i^* \in \{\{1\}, \{0,1\}\}, i = 1, 2, \ldots, m$, such that

$$\alpha(B_1^*, \ldots, B_m^*) \subset \left[k, \frac{n}{2}\right],$$

and

$$\sup_{\alpha(B_1, B_2, \ldots, B_m) \subset [k, \frac{n}{2}]} \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\} = \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i^*\}.$$

Then, a graph H^* with $V(H^*) = V(\mathcal{G})$ and $E(H^*) = \{e_i | e_i \in E(\mathcal{G}), B_i^* = \{1\}\}$ can be constructed. Thus, $H^* \in \mathcal{N}(k)$ and

$$\begin{aligned} \sup_{H \in \mathcal{N}(k)} \min_{e_i \in E(H)} \mathcal{M}\{\xi_i = 1\} &\geq \min_{e_i \in E(H^*)} \mathcal{M}\{\xi_i = 1\} \\ &= \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i^*\} \\ &= \sup_{\alpha(\mathcal{G}) \subset [k, +\infty)} \min_{1 \leq i \leq m} \mathcal{M}\{\xi_i \in B_i\}. \end{aligned}$$

That is, Inequality (4.3) holds.

Step 2: Next, we will prove that

$$\sup_{H\in\mathcal{N}(k)}\min_{e_i\in E(H)}\mathcal{M}\{\xi_i=1\}\leq\sup_{\alpha(\mathcal{G})\subset[k,+\infty)}\min_{1\leq i\leq m}\mathcal{M}\{\xi_i\in B_i\}. \tag{4.4}$$

As we know, there must exist a model $H^{'}\in\mathcal{N}(k)$, satisfying

$$\sup_{H\in\mathcal{N}(k)}\min_{e_i\in E(H)}\mathcal{M}\{\xi_i=1\}=\min_{e_i\in E(H^{'})}\mathcal{M}\{\xi_i=1\}.$$

Choose a series $\{B_1',B_2',\ldots,B_m'\}$ such that

$$B_i'=\begin{cases}\{1\}, & if \quad e_i\in E(H'),\\ \{0,1\}, & if \quad e_i\notin E(H').\end{cases}$$

Since $H'\in\mathcal{N}(k)$, then $\alpha(H')\geq k$. According to the choice of $\{B_i'\}$, it is clearly that

$$\alpha(B_1',\ldots,B_m')\subset\left[k,\frac{n}{2}\right],$$

and

$$\min_{e_i\in E(H')}\mathcal{M}\{\xi_i=1\}=\min_{1\leq i\leq m}\mathcal{M}\{\xi_i\in B_i'\}.$$

Then, we have

$$\sup_{\alpha(B_1,\ldots,B_m)\subset[k,\frac{n}{2}]}\min_{1\leq i\leq m}\mathcal{M}\{\xi_i\in B_i\}\geq\min_{1\leq i\leq m}\mathcal{M}\{\xi_i\in B_i'\}=\min_{e_i\in E(H')}\mathcal{M}\{\xi_i=1\}.$$

Thus, Inequality (4.4) holds immediately.

According to Inequalities (4.3) and (4.3), the proof is completed.

Let $\mathcal{G}$ be an uncertain graph with edge set $E=\{e_1,e_2,\ldots,e_m\}$, and indicator function $\xi=\{\xi_1,\xi_2,\ldots,\xi_m\}$. Theorem 4.15 shows that $\mathcal{M}\{\alpha(\mathcal{G})\geq k\}$ takes values in the set of $\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$. Thus, Li-Zhang-Peng [41] designed an algorithm for calculating the value of $\mathcal{M}\{\alpha(\mathcal{G})\geq k\}$.

Algorithm 4.4 (Matching Number Algorithm)

Step 1. Sort the set $\{\alpha_1,\alpha_2,\ldots,\alpha_m\}$ such that $1=\alpha_0\geq\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_m\geq\alpha_{m+1}=0$. Set $t=1$.

Step 2. A new uncertain graph $\mathcal{G}_t$ can be obtained by removing the edge e_i that satisfies $\alpha_i<\alpha_t, i=1,2,\ldots,m$.

Step 3. Denote the underlying graph of $\mathcal{G}_t$ by $\mathcal{G}_t^*$. Find a maximum matching in the graph $\mathcal{G}_t^*$, and calculate $\alpha(\mathcal{G}_t^*)$. Set $k_t=\alpha(\mathcal{G}_t^*)$.

Step 4. If $k_t\geq k$, stop. Otherwise, let $t=t+1$, if $t=m+1$, stop; if $t<m+1$, then go to **Step 2**.

In Step 4, if the algorithm is terminated by $t = m + 1$, then $\mathcal{M}\{\alpha(\mathcal{G}) \geq k\} = 0$; if the algorithm is terminated by $k_t \geq k$, then $\mathcal{M}\{\alpha(\mathcal{G}) \geq k\} = \alpha_t$.

In Step 3, in order to find a maximum matching in the graph $\mathcal{G}_t^*$, many maximum matching algorithms, such as Edmonds's matching algorithm (Edmonds [15]) and Gabow's matching algorithm (Gabow [19]), can be used.

Theorem 4.16 *(Li-Zhang-Peng [41]) For a given positive integer k, Algorithm 4.4 gives an exact value of $\mathcal{M}\{\alpha(\mathcal{G}) \geq k\}$.*

Proof: Step 1 shows that as the value of index t is decreases, the value of α_t is increases. If $t_1 > t_2$, according to Step 2, less edges are removed in the t_1-th iteration, and then uncertain graph $\mathcal{G}_{t_1}$ contains more edges. That is, $k_{t_1} \geq k_{t_2}$ if $t_1 > t_2$. In the following, the proof breaks down into the two cases.

Case 1: The algorithm is terminated by $t = m + 1$, which shows that $k_m < k$ in the m-th iteration. In fact, no edge will be removed in the m-th iteration. In other words, $\mathcal{G}_m = \mathcal{G}$, and $\mathcal{G}_m^* = \mathcal{G}^*$. In addition, the algorithm is terminated by $k_m = \alpha(\mathcal{G}_m^*) = \alpha(\mathcal{G}^*) < k$, which means that there exists no model H of $\mathcal{G}$ satisfying $\alpha(H) \geq k$. Thus, $\mathcal{M}\{\alpha(\mathcal{G}) \geq k\} = 0$.

Case 2: The algorithm is terminated by $k_t \geq k$, which shows that $k_{t-1} < k$ in the $t-1$-th iteration. It follows from the Algorithm 4.4 that $\alpha(\mathcal{G}_t^*) = k_t$, i.e., $\mathcal{G}_t^* \in \mathcal{N}(k_t)$. Based on Theorem 4.15 and Algorithm 4.4, we have

$$\mathcal{M}\{\alpha(\mathcal{G}) \geq k_t\} = \sup_{H \in \mathcal{N}(k_t)} \min_{e_i \in E(H)} \mathcal{M}\{\xi_i = 1\} \geq \min_{e_i \in E(\mathcal{G}_t^*)} \mathcal{M}\{\xi_i = 1\} = \alpha_t.$$

Since $k_t \geq k$, we have

$$\mathcal{M}\{\alpha(\mathcal{G}) \geq k\} \geq \mathcal{M}\{\alpha(\mathcal{G}) \geq k_t\} \geq \alpha_t.$$

If

$$\mathcal{M}\{\alpha(\mathcal{G}) \geq k\} > \alpha_t, \tag{4.5}$$

Algorithm 4.4 tells us that the value of $\mathcal{M}\{\alpha(\mathcal{G}) \geq k\}$ is located in the set $\{\alpha_1, \alpha_2, \ldots, \alpha_{t-1}\}$. Without loss of generality, we can assume that $\mathcal{M}\{\alpha(\mathcal{G}) \geq k\} = \alpha_{t-1}$.

According to Theorem 4.4, there exists a graph $H \in \mathcal{N}(k)$ such that

$$\mathcal{M}\{\alpha(\mathcal{G}) \geq k\} = \min_{e_i \in E(H)} \mathcal{M}\{\xi_i = 1\} = \alpha_{t-1}.$$

Furthermore, Algorithm 4.4 implies that graph $\mathcal{G}_{t-1}$ contains all the edges $e_i \in E(\mathcal{G})$ with $\alpha_i \geq \alpha_{t-1}$. That is, H is a spanning graph of $\mathcal{G}_{t-1}^*$. So

$$\alpha(H) \leq \alpha(\mathcal{G}_{t-1}^*) = k_{t-1} < k,$$

from which we can conclude $H \notin \mathcal{N}(k)$. Therefore, assumption (4.5) is not true, which means

$$\mathcal{M}\{\alpha(\mathcal{G}) \geq k\} = \alpha_t.$$

Thus, the proof is completed.

Example 4.6: Consider an uncertain graph $\mathcal{G}$ with 7 vertices and 10 edges shown in Figure 4.6. We will employ the proposed Algorithm 4.4 to calculate the uncertain measure that $\mathcal{G}$ is 3-edge matching, i.e., $\mathcal{M}\{\alpha(\mathcal{G}) \geq 3\}$.

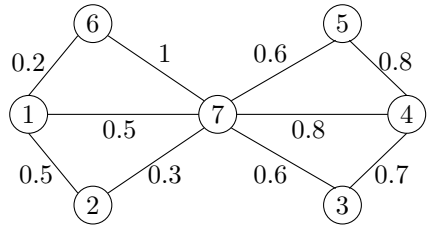


Figure 4.6: An uncertain graph $\mathcal{G}$ with 7 vertices and 10 edges

According to Theorem 4.15, the value of $\mathcal{M}\{\alpha(\mathcal{G}) \geq 3\}$ is located in the set of $\{1, 0.8, 0.7, 0.6, 0.5, 0.3, 0.2\}$.

In the first iteration, by removing the edges e_i that satisfy $\alpha_i < 1$ in the uncertain graph $\mathcal{G}$, a new uncertain graph $\mathcal{G}_1$ (see Figure 4.7) is obtained. Denote the underlying graph of $\mathcal{G}_1$ is $\mathcal{G}_1^*$, and we get $k_1 = \alpha(\mathcal{G}_1^*) = 1$. According to Algorithm 4.4, we should continue the iteration.

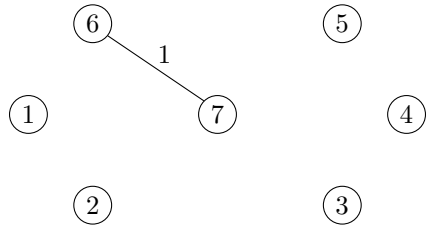


Figure 4.7: Uncertain graph $\mathcal{G}_1$

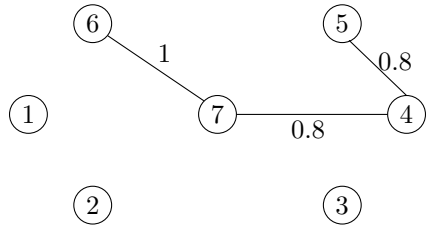


Figure 4.8: Uncertain graph $\mathcal{G}_2$

In the second iteration, after removing the edges e_i that satisfy $\alpha_i < 0.8$ in the uncertain graph $\mathcal{G}$, a new uncertain graph $\mathcal{G}_2$ is obtained, which is

shown in Figure 4.8. Denote the underlying graph of $\mathcal{G}_2$ is $\mathcal{G}_2^*$. It is clearly that $k_2 = \alpha(\mathcal{G}_2^*) = 2$. Thus, we should continue the iteration.

In the third iteration, we obtain a new uncertain graph $\mathcal{G}_3$ (see Figure 4.9) by removing the edges e_i with $\alpha_i < 0.7$ in the uncertain graph $\mathcal{G}$. Denote the underlying graph of $\mathcal{G}_3$ as $\mathcal{G}_3^*$. We have $k_3 = \alpha(\mathcal{G}_3^*) = 2$.

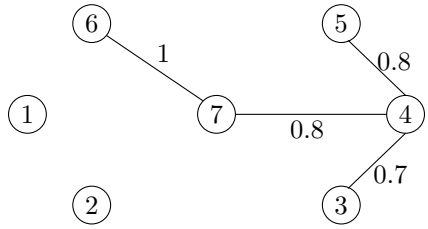


Figure 4.9: Uncertain graph $\mathcal{G}_3$

In the fourth iteration, we obtain a new uncertain graph $\mathcal{G}_4$ shown in Figure 4.10 by removing the edges e_i with $\alpha_i < 0.6$ in the uncertain graph $\mathcal{G}$. It is easy to verify that, $k_4 = \alpha(\mathcal{G}_4^*) = 2$, where $\mathcal{G}_4^*$ is the underlying graph of $\mathcal{G}_4$.

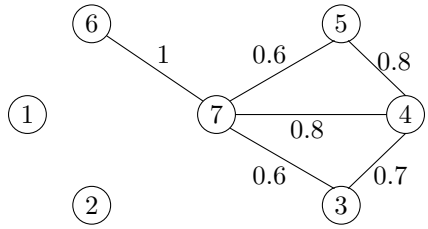


Figure 4.10: Uncertain graph $\mathcal{G}_4$

After the fifth iteration, a new uncertain graph $\mathcal{G}_5$ is obtained, which is shown in Figure 4.11. Obviously, $k_5 = \alpha(\mathcal{G}_5^*) = 3$, where $\mathcal{G}_5^*$ is the underlying graph of $\mathcal{G}_5$. According to Algorithm 4.4, the iteration can be terminated, and $\mathcal{M}\{\alpha(\mathcal{G}) \geq 3\} = 0.5$.

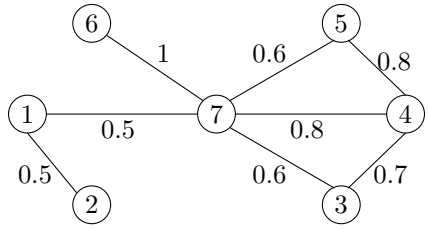


Figure 4.11: Uncertain graph $\mathcal{G}_5$

In addition, according to Algorithm 4.4, we can also obtain $\mathcal{M}\{\alpha(\mathcal{G}) \geq 1\} = 1$, $\mathcal{M}\{\alpha(\mathcal{G}) \geq 2\} = 0.8$, and $\mathcal{M}\{\alpha(\mathcal{G}) \geq k\} = 0$ for any positive integer with $k \geq 4$. It follows from the duality axiom that, we have

$$\mathcal{M}\{\alpha(\mathcal{G}) \leq 0\} = 0, \qquad \mathcal{M}\{\alpha(\mathcal{G}) \leq 1\} = 0.2,$$
$$\mathcal{M}\{\alpha(\mathcal{G}) \leq 2\} = 0.5, \qquad \mathcal{M}\{\alpha(\mathcal{G}) \leq 3\} = 1.$$

Furthermore, the uncertainty distribution of $\alpha(\mathcal{G})$ is obtained as follows,

$$\Phi(x) = \begin{cases} 0, & \text{if } x < 1, \\ 0.2, & \text{if } 1 \leq x < 2, \\ 0.5, & \text{if } 2 \leq x < 3, \\ 1, & \text{if } x \geq 3. \end{cases}$$

Chapter 5

Uncertain Network Optimization

Uncertain network optimization is a type of optimization method involving uncertain variables. This chapter will introduce the uncertain network optimization by some problems with uncertain variables.

5.1 Uncertain Chinese Postman Problem

It is well known that the motivation for the Chinese postman problem came from the job of a postman. In his job, the postman picks up mail at the post office, delivers it, and then ends up at the same place where he began his route. The postman, of course, must cover each street in his area at least once. Based on this condition, the postman wishes to seek the shortest way to complete his assigned route of streets. This problem was first considered by a Chinese mathematician, Kuan [38].

In general, a deterministic network is denoted as $N = (V, A, w)$, where $V = \{1, 2, \ldots, n\}$ is a finite set of vertices, $A = \{(i, j)|i, j \in V\}$ is the set of edges, and $w = \{w_{ij}|(i, j) \in A\}$ is the set of edge weights. Here, w_{ij} denotes the length from i to j. It is easy to see that the values of w_{ij} in the deterministic network are crisp values. A route of N is a cycle that traverses each edge of N at least once. The Chinese postman problem is just to find a route with the shortest length, which is called the shortest route. The length of the shortest route is denoted as f_{SR}. Clearly, f_{SR} is a function of w. Given w, $f_{SR}(w)$ can be obtained by using Edmonds and Johnson's algorithm (Edmonds-Johnson [16]). In the following, Example 5.1 illustrates function $f_{SR}(w)$ with respect to w.

Example 5.1: Let us consider the Chinese postman problem on the network $N = (V, A, w)$, which is shown in Figure 5.1.

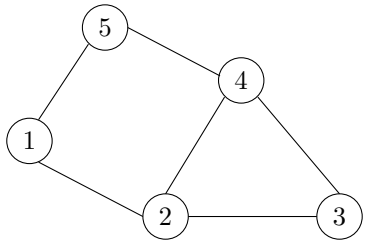


Figure 5.1: Network $N = (V, A, w)$

Suppose that $w_{ij} = w_{ji}$. When $w = (w_{12}, w_{15}, w_{23}, w_{24}, w_{34}, w_{45}) = (1, 2, 1,\ 1, 2, 3)$, the shortest route is $1 \to 5 \to 4 \to 3 \to 2 \to 4 \to 2 \to 1$; that is, $f_{SR}(w) = w_{15} + w_{54} + w_{43} + w_{32} + 2w_{24} + w_{21} = 11$. When $w = (w_{12}, w_{15}, w_{23}, w_{24}, w_{34}, w_{45}) = (1, 2, 1, 3, 1, 3)$, the shortest route is $1 \to 5 \to 4 \to 3 \to 2 \to 3 \to 4 \to 2 \to 1$; that is, $f_{SR}(w) = w_{15} + w_{54} + w_{43} + 2w_{32} + w_{34} + w_{42} + w_{21} = 13$.

In an uncertain network, the lengths of edges are not precisely known and they are specified as uncertain variables. That is, each length w_{ij} is replaced by a nonnegative uncertain variable ξ_{ij}. We can denote the network with uncertain weight as $N = (V, A, \xi)$, where $\xi = \{\xi_{ij}|(i, j) \in A\}$. It is shortest route length is $f_{SR}(\xi)$. Obviously, $f_{SR}(\xi)$ is an uncertain variable. Denote $\Psi(x)$ as the uncertainty distribution of $f_{SR}(\xi)$. How can we get $\Psi(x)$? The following subsection will answer this question.

Uncertainty Distribution $\Psi(x)$

As we know, in a deterministic network $N(V, A, w)$, the shortest route length $f_{SR}(w)$ is strictly increasing with respect to w, where $w = \{w_{ij}|(i, j) \in A\}$. That is, if some values of w_{ij} increase, the value of $f_{SR}(w)$ may not increase; if all the values of w_{ij} increase, then, the value of $f_{SR}(w)$ must increase. According to Theorem 2.4, we can easily obtain the inverse uncertainty distribution of $f_{SR}(\xi)$.

Theorem 5.1 *Let $N = (V, A, \xi)$ be an uncertain network. If ξ_{ij} have regular uncertainty distributions Φ_{ij}, then the inverse uncertainty distribution of $f_{SR}(\xi)$ is determined by*

$$\Psi^{-1}(\alpha) = f_{SR}(\Phi_{ij}^{-1}(\alpha)|(i, j) \in A).$$

Example 5.2: Still consider the Chinese postman problem on the network $N = (V, A, \xi)$ presented in Figure 5.1. Assume that ξ_{ij} are independent zigzag uncertain variables, $i, j = 1, 2, \ldots, 5$, respectively, which are shown as follows,

$$\xi_{12} \sim \mathcal{Z}(1, 2, 4), \qquad \xi_{15} \sim \mathcal{Z}(2, 4, 5), \qquad \xi_{23} \sim \mathcal{Z}(1, 3, 4),$$

$$\xi_{24} \sim \mathcal{Z}(1,3,6), \qquad \xi_{34} \sim \mathcal{Z}(2,4,5), \qquad \xi_{45} \sim \mathcal{Z}(1,4,7).$$

When $\alpha = 0.8$, it is easy to get $\Phi_{12}^{-1}(0.8) = 3.2$, $\Phi_{15}^{-1}(0.8) = 4.6$, $\Phi_{23}^{-1}(0.8) = 3.6$, $\Phi_{24}^{-1}(0.8) = 4.8$, $\Phi_{34}^{-1}(0.8) = 4.6$, $\Phi_{45}^{-1}(0.8) = 5.8$, as shown in Figure 5.2. Then, Edmonds and Johnson's algorithm tells us that

$$\begin{aligned}\Psi^{-1}(0.8) &= f_{SR}(\Phi_{12}^{-1}(0.8), \Phi_{15}^{-1}(0.8), \Phi_{23}^{-1}(0.8), \Phi_{24}^{-1}(0.8), \Phi_{34}^{-1}(0.8), \Phi_{45}^{-1}(0.8)) \\ &= f_{SR}(3.2, 4.6, 3.6, 4.8, 4.6, 5.8) \\ &= 31.4.\end{aligned}$$

That is, $\mathcal{M}\{f_{SR}(\xi) \leq 31.4\} = \Psi(31.4) = 0.8$.

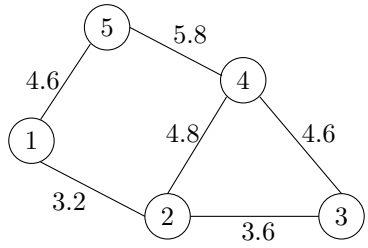


Figure 5.2: Network $N = (V, A, \xi)$ when $\alpha = 0.8$

When $\alpha = 0.9$, it is easy to get $\Phi_{12}^{-1}(0.9) = 3.6$, $\Phi_{15}^{-1}(0.9) = 4.8$, $\Phi_{23}^{-1}(0.9) = 3.8$, $\Phi_{24}^{-1}(0.9) = 5.4$, $\Phi_{34}^{-1}(0.9) = 4.8$, $\Phi_{45}^{-1}(0.9) = 6.4$, as shown in Figure 5.3. Then, Edmonds and Johnson's algorithm tells us that

$$\begin{aligned}\Psi^{-1}(0.9) &= f_{SR}(\Phi_{12}^{-1}(0.9), \Phi_{15}^{-1}(0.9), \Phi_{23}^{-1}(0.9), \Phi_{24}^{-1}(0.9), \Phi_{34}^{-1}(0.9), \Phi_{45}^{-1}(0.9)) \\ &= f_{SR}(3.6, 4.8, 3.8, 5.4, 4.8, 6.4) \\ &= 34.2.\end{aligned}$$

That is, $\mathcal{M}\{f_{SR}(\xi) \leq 34.2\} = \Psi(34.2) = 0.9$.

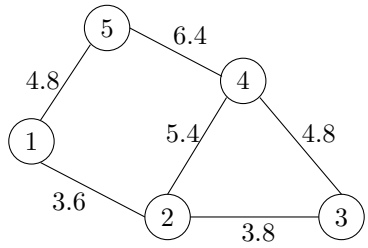


Figure 5.3: Network $N = (V, A, \xi)$ when $\alpha = 0.9$

Repeating this process, for any $\alpha \in (0, 1)$, we can obtain $\Psi^{-1}(\alpha)$. Thus we can get the uncertainty distribution $\Psi(x)$ by interpolation.

Decision Criteria

To obtain the shortest route in an uncertain network, some addition criteria should be presented.

For convenience, a route in the uncertain network $N = (V, A, \xi)$ is denoted as $\widetilde{R}$. To describe the Chinese postman problem by mathematical model, the decision variables x_{ij} are defined as follows,

$$x_{ij} = \begin{cases} 1, & \text{if there exists an edge from } i \text{ to } j \text{ in route } \widetilde{R} \\ 0, & \text{otherwise} \end{cases}$$

for any $(i, j) \in A$. Then the length of $\widetilde{R}$ is

$$l(\widetilde{R}) = \sum_{(i,j)\in A} x_{ij}\xi_{ij}. \tag{5.1}$$

Thus, $\widetilde{R}$ is a route if and only if

$$\begin{cases} \sum\limits_{j:(i,j)\in A} x_{ij} - \sum\limits_{j:(j,i)\in A} x_{ji} = 0 \quad \text{for } i \in V, \\ x_{ij} + x_{ji} \geq 1 \quad \text{for } (i, j) \in A, \\ x_{ij} = \{0, 1\} \quad \text{for } (i, j) \in A. \end{cases}$$

Remark 5.1: The first constraint requires that the route is cyclical, and the second constraint requires that the route traverses each edge of N at least once.

Then, for uncertain network $N = (V, A, \xi)$, a key problem is how to obtain the shortest optimal route. To solve the problem, Zhang and Peng [72] provided three approaches called expected shortest route, α-shortest route and best shortest route.

Definition 5.1 *(Zhang-Peng [72]) Let $N = (V, A, \xi)$ be an uncertain network, and $\widetilde{R}^*$ a route. Then $\widetilde{R}^*$ is called the expected shortest route (ESR) if*

$$E[l(\widetilde{R}^*)] \leq E[l(\widetilde{R})]$$

holds for any route $\widetilde{R}$.

Definition 5.2 *(Zhang-Peng [72]) Let $N = (V, A, \xi)$ be an uncertain network, and $\widetilde{R}^*$ a route. Then $\widetilde{R}^*$ is called α-shortest route (α-SR) if*

$$\min\{W | \mathcal{M}\{l(\widetilde{R}^*) \leq W\} \geq \alpha\} \leq \min\{W | \mathcal{M}\{l(\widetilde{R}) \leq W\} \geq \alpha\}$$

for any route $\widetilde{R}$, and α is a predetermined confidence level.

The meaning of the α-shortest route can be summarized as follows. Given an $\alpha \in (0,1)$, we hope to get the smallest length W and a route $\widetilde{R}^*$, where uncertain variable $l(\widetilde{R}^*)$ is less than or equal to W with confidence level α.

Definition 5.3 *(Zhang-Peng [72]) Let $N = (V, A, \xi)$ be an uncertain network, and $\widetilde{R}^*$ a route. Then $\widetilde{R}^*$ is called best shortest route (BSR) if*

$$\mathcal{M}\{l(\widetilde{R}^*) \leq W\} \geq \mathcal{M}\{l(\widetilde{R}) \leq W\}$$

for all route $\widetilde{R}$, where W is a predetermined length.

For a given length W, best shortest route R^* is the shortest route which is less than or equal to W with the largest confidence level.

Expected Shortest Route (ESR)

This subsection provides a method to find the ESR. Firstly, based on the linearity of expectation, we have the following equivalent definition of the ESR as follows.

Theorem 5.2 *(Equivalent Definition of ESR) Let $N(V, A, \xi)$ be an uncertain network. If ξ_{ij} are independent uncertain variables. Then, a route $\widetilde{R}^*$ is an ESR if and only if*

$$\sum_{(i,j)\in\widetilde{R}^*} E[\xi_{ij}] \leq \sum_{(i,j)\in\widetilde{R}} E[\xi_{ij}]$$

holds for any route $\widetilde{R}$.

Proof: Since ξ_{ij} are independent uncertain variables, following from Theorem 2.12, then

$$E[l(\widetilde{R}^*)] = E\left[\sum_{(i,j)\in\widetilde{R}^*} \xi_{ij}\right] = \sum_{(i,j)\in\widetilde{R}^*} E[\xi_{ij}].$$

Similarly,

$$E[l(\widetilde{R})] = \sum_{(i,j)\in\widetilde{R}} E[\xi_{ij}].$$

Therefore, we get that $E[l(\widetilde{R}^*)] \leq E[l(\widetilde{R})]$ holds if and only if

$$\sum_{(i,j)\in\widetilde{R}^*} E[\xi_{ij}] \leq \sum_{(i,j)\in\widetilde{R}} E[\xi_{ij}]$$

is true. According to Definition 5.1, the theorem is proved.

Theorem 5.3 *(Zhang-Peng [72]) Let $N(V, A, \xi)$ be an uncertain network. If ξ_{ij} are independent uncertain variables. Then, the expected shortest route is just the shortest route of $N(\overline{V}, \overline{A}, w)$, where $\overline{V} = V, \overline{A} = A$, and $w_{ij} = E[\xi_{ij}]$.*

Proof: According to Definition 5.1, the ESR $\widetilde{R}^*$ is the optimal solution to the following uncertain programming model:

$$\begin{cases} \min\limits_{x_{ij}} & E\left[\sum\limits_{(i,j)\in A} x_{ij}\xi_{ij}\right] & \\ s.t. & \sum\limits_{j:(i,j)\in A} x_{ij} - \sum\limits_{j:(j,i)\in A} x_{ji} = 0 & \text{for } i \in V \\ & x_{ij} + x_{ji} \geq 1 & \text{for } (i,j) \in A \\ & x_{ij} = \{0,1\} & \text{for } (i,j) \in A. \end{cases} \tag{5.2}$$

Since ξ_{ij} are independent uncertain variables, it follows from the linearity of expected value operator of uncertain variable that model (5.2) can be transformed into the following deterministic one:

$$\begin{cases} \min\limits_{x_{ij}} & \sum\limits_{(i,j)\in A} x_{ij}E[\xi_{ij}] & \\ s.t. & \sum\limits_{j:(i,j)\in A} x_{ij} - \sum\limits_{j:(j,i)\in A} x_{ji} = 0 & \text{for } i \in V \\ & x_{ij} + x_{ji} \geq 1 & \text{for } (i,j) \in A \\ & x_{ij} = \{0,1\} & \text{for } (i,j) \in A. \end{cases} \tag{5.3}$$

In fact, the model (5.3) describes the shortest route in network $N = (V, A, w)$, where $\overline{V} = V, \overline{A} = A$, and $w_{ij} = E[\xi_{ij}]$. Thus, the theorem is proved.

Theorem 5.3 provides an effective method to obtain ESR in uncertain network $N = (V, A, \xi)$. According to Theorem 5.3, if we want to find the ESR of a given uncertain network, we can find the shortest route of its corresponding deterministic network by using an existing algorithm for the classical Chinese postman problem which can be solved in polynomial time. The method can be summarized as follows,

Algorithm 5.1 (Algorithm for ESR)

Step 1. Calculate $E[\xi_{ij}]$ for each uncertain length ξ_{ij} in $N = (V, A, \xi)$.

Step 2. Construct a deterministic network $N = (V, A, w)$, whose edge weights $w_{ij} = E[\xi_{ij}]$.

Step 3. Employ Edmonds and Johnson's algorithm to get the shortest route in the network $N = (V, A, w)$.

The shortest route we get in Step 3 is just the ESR in $N = (V, A, \xi)$.

α-shortest Route (α-SR)

In this subsection, let us consider the α-SR in Definition 5.2. According to the inverse uncertainty distribution of uncertain variables, we propose some equivalent definitions of α-SR. Moreover, we also present an apporach to find the α-SR.

Theorem 5.4 *Let $N(V, A, \xi)$ be an uncertain network, and ξ_{ij} have regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \ldots, n$, respectively. Then, for any route $\widetilde{R}$, the length is an uncertain variable with inverse uncertainty distribution*

$$\Psi_{\widetilde{R}}^{-1}(\alpha) = \sum_{(i,j)\in\widetilde{R}} \Phi_{ij}^{-1}(\alpha), \quad \alpha \in (0, 1).$$

Proof: Since ξ_{ij}, $i, j = 1, 2, \ldots, n$, are independent uncertain variables with regular uncertainty distributions, and α is a predetermined confidence level, this theorem follows from equation (5.1) and Theorem 2.4 immediately.

By utilizing the inverse uncertainty distribution of the length of a route, we can obtain an equivalent definition of α-SR as follows.

Theorem 5.5 *(Equivalent Definition I of α-SR) Let $N(V, A, \xi)$ be an uncertain network, ξ_{ij} have regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \ldots, n$, respectively. Then, a route $\widetilde{R}^*$ is an α-SR if and only if*

$$\Psi_{\widetilde{R}^*}^{-1}(\alpha) \leq \Psi_{\widetilde{R}}^{-1}(\alpha)$$

holds for any route $\widetilde{R}$, where $\Psi_{\widetilde{R}^*}^{-1}$ and $\Psi_{\widetilde{R}}^{-1}$ are the inverse uncertainty distributions of the lengths $l(\widetilde{R}^*)$ and $l(\widetilde{R})$, respectively.

Proof: According to Theorem 5.4, it is clear that $l(\widetilde{R})$ is an uncertain variable with regular uncertainty distribution. Consequently, following Definition 2.3, we have

$$\min\{W | \mathcal{M}\{l(\widetilde{R}) \leq W\} \geq \alpha\} = \min\{W | \Psi_{\widetilde{R}}(W) \geq \alpha\} = \Psi_{\widetilde{R}}^{-1}(\alpha).$$

Similarly, we also have

$$\min\{W | \mathcal{M}\{l(\widetilde{R}^*) \leq W\} \geq \alpha\} = \Psi_{\widetilde{R}^*}^{-1}(\alpha).$$

Thus, the theorem holds.

Theorem 5.6 *(Equivalent Definition II of α-SR) Let $N(V, A, \xi)$ be an uncertain network, ξ_{ij} have regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \ldots, n$, respectively. Then, a route $\widetilde{R}^*$ is an α-SR if and only if*

$$\sum_{(i,j)\in\widetilde{R}^*} \Phi_{ij}^{-1}(\alpha) \leq \sum_{(i,j)\in\widetilde{R}} \Phi_{ij}^{-1}(\alpha)$$

holds for any route $\widetilde{R}$, where $\Psi_{\widetilde{R}^*}^{-1}$ and $\Psi_{\widetilde{R}}^{-1}$ are the inverse uncertainty distributions of the lengths $l(\widetilde{R}^*)$ and $l(\widetilde{R})$, respectively.

Proof: According to Theorem 5.4, for the route $\widetilde{R}^*$ and any given route $\widetilde{R}$, we have

$$\Psi_{\widetilde{R}^*}^{-1}(\alpha) = \sum_{(i,j)\in\widetilde{R}^*} \Phi_{ij}^{-1}(\alpha),$$

and

$$\Psi_{\widetilde{R}}^{-1}(\alpha) = \sum_{(i,j)\in\widetilde{R}} \Phi_{ij}^{-1}(\alpha).$$

Then this theorem follows from Theorem 5.5 immediately.

Theorem 5.7 *(Zhang-Peng [72]) Let $N(V, A, \xi)$ be an uncertain network, ξ_{ij} have regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \ldots, n$, respectively. Then, the α-SR is just the shortest route of $\overline{N} = (\overline{V}, \overline{A}, w)$, where $\overline{V} = V, \overline{A} = A$, and $w_{ij} = \Phi_{ij}^{-1}(\alpha)$.*

Proof: According to Definition 5.2, α-SR is the optimal solution to the following programming model:

$$\begin{cases} \min & W \\ s.t. & \mathcal{M}\left\{\sum\limits_{(i,j)\in A} x_{ij}\xi_{ij} \le W\right\} \ge \alpha, \\ & \sum\limits_{j:(i,j)\in A} x_{ij} - \sum\limits_{j:(j,i)\in A} x_{ji} = 0 \quad \text{for } i \in V \\ & x_{ij} + x_{ji} \ge 1 \quad \text{for } (i,j) \in A \\ & x_{ij} = \{0, 1\} \quad \text{for } (i,j) \in A. \end{cases} \tag{5.4}$$

Since ξ_{ij} have regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \ldots, n$, respectively, model (5.4) can be equivalently transformed into the following deterministic one:

$$\begin{cases} \min & W \\ s.t. & \sum\limits_{(i,j)\in A} x_{ij}\Phi_{ij}^{-1}(\alpha) \le W, \\ & \sum\limits_{j:(i,j)\in A} x_{ij} - \sum\limits_{j:(j,i)\in A} x_{ji} = 0 \quad \text{for } i \in V \\ & x_{ij} + x_{ji} \ge 1 \quad \text{for } (i,j) \in A \\ & x_{ij} = \{0, 1\} \quad \text{for } (i,j) \in A. \end{cases} \tag{5.5}$$

It is easy to verify that model (5.5) is equivalent to

$$\begin{cases} \min & \displaystyle\sum_{(i,j)\in A} x_{ij}\Phi_{ij}^{-1}(\alpha) & \\ s.t. & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = 0 & \text{for } i \in V \\ & x_{ij} + x_{ji} \geq 1 & \text{for } (i,j) \in A \\ & x_{ij} = \{0,1\} & \text{for } (i,j) \in A. \end{cases} \tag{5.6}$$

Note that, model (5.6) describes the shortest route in the network $N = (V, A, w)$, where $\overline{V} = V, \overline{A} = A$, and $w_{ij} = \Phi_{ij}^{-1}(\alpha)$. Hence the theorem is proved.

Theorem 5.7 provides an effective method to obtain α-SR in uncertain network $N = (V, A, \xi)$. The method can be summarized as follows,

Algorithm 5.2 (Algorithm for $\alpha - SR$)

Step 1. Calculate $\Phi_{ij}^{-1}(\alpha)$, for each length ξ_{ij} in $N = (V, A, \xi)$.

Step 2. Construct a deterministic network $N = (V, A, w)$, whose edge weights $w_{ij} = \Phi_{ij}^{-1}(\alpha)$.

Step 3. Employ Edmonds and Johnson's algorithm to get the shortest route in the network $N = (V, A, w)$.

The shortest route we get in Step 3 is just the α-SR in $N = (V, A, \xi)$.

Best Shortest Route (BSR)

According to Definition 5.3, length of the BSR is less than the given length W with the greatest chance. Hence, a measure-chance programming to find the BSR can be formulated as follows,

$$\begin{cases} \max & \mathcal{M}\left\{\displaystyle\sum_{(i,j)\in A} x_{ij}\xi_{ij} \leq W\right\} & \\ s.t. & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = 0 & \text{for } i \in V \\ & x_{ij} + x_{ji} \geq 1 & \text{for } (i,j) \in A \\ & x_{ij} = \{0,1\} & \text{for } (i,j) \in A. \end{cases} \tag{5.7}$$

In the following, we will discuss the relation between the α-SR and the BSR in an uncertain network. Then we provide an effective approach to obtain the BSR.

Theorem 5.8 *Let $N(V, A, \xi)$ be an uncertain network, and ξ_{ij} have regular uncertainty distributions Φ_{ij}, $(i,j) \in A$. Given a predetermined length W, the BSR is just the α-SR of $N(V, A, \xi)$, where $\alpha = \Psi(W)$.*

Proof: As we know, the BSR is the optimal solution to the model (5.7). Set

$$\mathcal{M}\left\{\sum_{(i,j)\in A} x_{ij}\xi_{ij} \le W\right\} \ge \alpha,$$

then model (5.7) is equivalent to the following model:

$$\begin{cases} \max & \alpha \\ s.t. & \mathcal{M}\left\{\sum_{(i,j)\in A} x_{ij}\xi_{ij} \le W\right\} \ge \alpha \\ & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = 0 \quad \text{for } i \in V \\ & x_{ij} + x_{ji} \ge 1 \quad \text{for } (i,j) \in A \\ & x_{ij} = \{0,1\} \quad \text{for } (i,j) \in A. \end{cases} \tag{5.8}$$

Denote (x, α) as the solution of model (5.8). Since each ξ_{ij}, $(i,j) \in A$, has a regular uncertainty distribution Φ_{ij}, the first constraint of model (5.8) can be reformulated as

$$\sum_{(i,j)\in A} x_{ij}\Phi_{ij}^{-1}(\alpha) \le W.$$

Thus, model (5.8) can be rebuilt as

$$\begin{cases} \max & \alpha \\ s.t. & \sum_{(i,j)\in A} x_{ij}\Phi_{ij}^{-1}(\alpha) \le W \\ & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = 0 \quad \text{for } i \in V \\ & x_{ij} + x_{ji} \ge 1 \quad \text{for } (i,j) \in A \\ & x_{ij} = \{0,1\} \quad \text{for } (i,j) \in A. \end{cases} \tag{5.9}$$

Set $\alpha' = \Psi(W)$. Since Ψ is a regular distribution, then $\Psi^{-1}(\alpha') = W$. For the confidence level α', according to Theorem 5.7, we can obtain the α'-SR x' of network (V, A, ξ). That is, x' is the optimal solution of model (5.5). Thus, x' and α' satisfy the following constraints

$$\begin{cases} \sum_{(i,j)\in A} x_{ij}\Phi_{ij}^{-1}(\alpha) \le W \\ \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = 0 \quad \text{for } i \in V \\ x_{ij} + x_{ji} \ge 1 \quad \text{for } (i,j) \in A \\ x_{ij} = \{0,1\} \quad \text{for } (i,j) \in A. \end{cases}$$

In other words, (x', α') is a feasible solution to model (5.9), and α' is the objective. In fact, α' is also the optimal objective.

Assume that (x'', α'') is a feasible solution of model (5.9) such that $\alpha'' > \alpha'$. As we know, the inverse distribution Ψ^{-1} is strictly increasing since Ψ is a regular uncertainty distribution. That is,

$$\Psi^{-1}(\alpha'') > \Psi^{-1}(\alpha') = W.$$

More precisely,

$$f_{SR}(\Phi_{ij}^{-1}(\alpha'')|(i,j) \in A) > f_{SR}(\Phi_{ij}^{-1}(\alpha')|(i,j) \in A) = W,$$

which means that, for any route R,

$$\sum_{(i,j)\in R} \Phi_{ij}^{-1}(\alpha'') \geq f_{SR}(\Phi_{ij}^{-1}(\alpha'')|(i,j) \in A) > W, \tag{5.10}$$

since $f_{SR}(\Phi_{ij}^{-1}(\alpha'')|(i,j) \in A)$ is the shortest length in network $\overline{N} = (\overline{V}, \overline{A}, w)$, where $\overline{V} = V, \overline{A} = A$, and $w_{ij} = \Phi_{ij}^{-1}(\alpha'')$.

Obviously, inequality (5.10) leads to a contradiction with the constraints in model (5.9). It shows that $\alpha' = \Psi(W)$ is the optimal objective value, and then x' is the optimal solution of model (5.9). Thus, the theorem is proved.

In uncertain network $N = (V, A, \xi)$, given a predetermined length W, the BSR can be obtained by the following algorithm.

Algorithm 5.3 (Algorithm for BSR)

Step 1. Obtain the uncertainty distribution Ψ by interpolation method.

Step 2. Set $\alpha = \Psi(W)$.

Step 3. Search the α-SR under the confidence level α.

According to Theorem 5.8, the α-SR we get in Step 3 is simply the BSR in $N = (V, A, \xi)$ that we want.

Numerical Examples

Now, we give some numerical examples to illustrate the efficiency of the models and algorithms as presented above for the uncertain Chinese postman problem. We consider an uncertain network $N = (V, A, \xi)$ with 7 vertices and 11 edges, depicted in Figure 5.4. The lengths ξ_{ij} of edge $(i, j) \in A$, which are supposed to be independent zigzag uncertain variables, are listed in the second column of Table 5.1. Suppose that $\xi_{ij} = \xi_{ji}$.

To obtain an expected optimal decision, the expected values $E[\xi_{ij}]$ of the lengths ξ_{ij} are listed in the third column of Table 5.1. As Algorithm 5.1 indicates, we should first construct a deterministic network $N = (V, A, w)$, whose edge weights $w_{ij} = E[\xi_{ij}]$. We employ Edmonds and Johnson's algorithm to

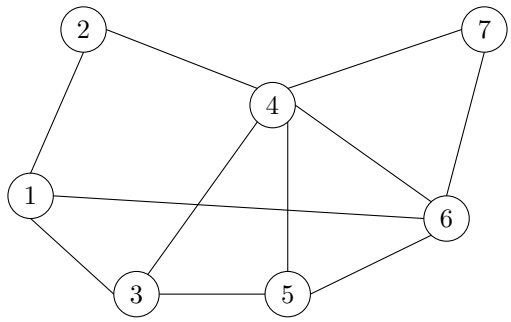


Figure 5.4: Uncertain network $N = (V, A, \xi)$

Table 5.1: List of ξ_{ij}

edge (i, j)	ξ_{ij}	$E[\xi_{ij}]$	$\Phi_{ij}^{-1}(0.8)$	$\Phi_{ij}^{-1}(0.95)$
$(1, 2)$	$\mathcal{Z}(5, 8, 10)$	7.75	9.2	9.8
$(1, 3)$	$\mathcal{Z}(4, 6, 9)$	6.25	7.8	8.7
$(1, 6)$	$\mathcal{Z}(12, 16, 18)$	15.5	17.2	17.8
$(2, 4)$	$\mathcal{Z}(6, 8, 13)$	8.75	11	12.5
$(3, 4)$	$\mathcal{Z}(6, 9, 14)$	9.5	12	13.5
$(3, 5)$	$\mathcal{Z}(4, 7, 9)$	6.75	8.2	8.8
$(4, 5)$	$\mathcal{Z}(7, 8, 10)$	8.25	9.2	9.8
$(4, 6)$	$\mathcal{Z}(8, 10, 13)$	10.25	11.8	12.7
$(4, 7)$	$\mathcal{Z}(10, 12, 15)$	12.25	13.8	14.7
$(5, 6)$	$\mathcal{Z}(4, 6, 9)$	6.25	7.8	8.7
$(6, 7)$	$\mathcal{Z}(5, 8, 9)$	7.5	8.6	8.9

get the expected optimal decision is: $x_{13} = x_{16} = x_{21} = x_{31} = x_{34} = x_{42} = x_{45} = x_{47} = x_{53} = x_{54} = x_{64} = x_{65} = x_{76} = 1$, which means that the ESR is

$$R_1 : 1 \to 3 \to 4 \to 5 \to 3 \to 1 \to 6 \to 5 \to 4 \to 7 \to 6 \to 4 \to 2 \to 1,$$

and the length of R_1 is 113.5.

Given $\alpha = 0.8$, we want to search the 0.8-SR. Algorithm 5.2 shows that we should first calculate $\Phi_{ij}^{-1}(0.8)$, which are listed in the fourth column of Table 5.1. Then, a deterministic network $N = (V, A, w)$, whose edge weights $w_{ij} = \Phi_{ij}^{-1}(0.8)$ can be constructed. By employing the Edmonds and Johnson's algorithm to the deterministic network N, the optimal decision is: $x_{13} = x_{16} = x_{21} = x_{31} = x_{35} = x_{42} = x_{43} = x_{45} = x_{54} = x_{56} = x_{64} = x_{67} = x_{74} = 1$, which means that the 0.8-SR is

$$R_2 : 1 \to 3 \to 1 \to 6 \to 4 \to 3 \to 5 \to 4 \to 5 \to 6 \to 7 \to 4 \to 2 \to 1,$$

and the length of R_2 is 133.6. That is, if we choose the paths as R_2, the total

length will be less then 133.6 with belief degree 80%. In other words, 133.6 is the minimum total length that can be achieved under the chance constraint $\alpha = 0.8$.

Given $\alpha = 0.95$, the values of $\Phi_{ij}^{-1}(0.95)$ are listed in the fifth column of Table 5.1. By the same approach, we can obtain the 0.95-SR whose total length is 144.4. Moreover, 144.4 is the minimum total length that can be achieved under the chance constraint $\alpha = 0.95$. In addition, for different α, we obtain a different total length. The results are shown in Table 5.2. Repeating this process, the distribution function Ψ of the shortest route $f_{SR}(\xi)$ can be obtained, which is shown in Figure 5.5. Obviously, as the confidence level α increases, the total length will increase. As we know, the greater α is, the more conservative the plan is.

Table 5.2: The minimum total length W for different α

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.95
W	88	94	100	106	112	119.2	126.4	133.6	140.8	144.4

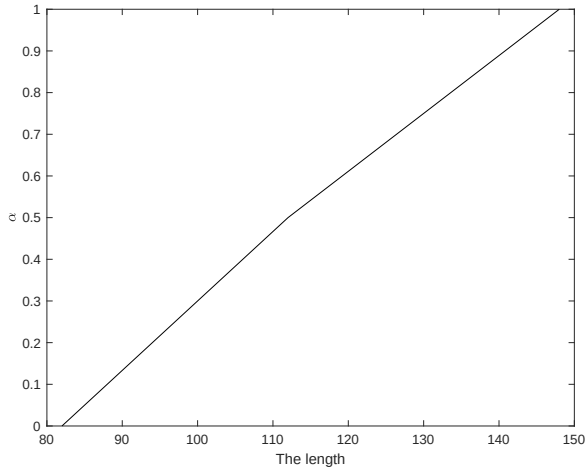


Figure 5.5: Uncertainty distribution of f_{SR}

After obtaining the distribution function Ψ, then we can get the BSR. Given a predetermined length $W = 141$, to obtain the BSR, we should first calculate $\alpha = \Psi(141)$ according to Theorem 5.8. From Table 5.2, we have $\Psi(141) \approx 0.9$. That is, $\alpha = 0.9$. Then, the BSR we want to obtain is simply the 0.9-SR.

5.2 Uncertain Assignment Problem

A typical optimal assignment problem is to allocate n jobs to n workers so that the total profit is maximized. For convenience, the profits made by n workers on n jobs can be shown by a profit matrix P,

$$P = \begin{pmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{pmatrix}$$

where p_{ij} indicates the profit made by worker i on job j.

Clearly, the optimal profit is a function of P, which is denoted as $f(P)$. Given P, the optimal assignment problem can be solved by some effective algorithms, such as the Kuhn-Munkres algorithm (Kuhn [39], Munkres [53]).

Example 5.3: Assume there are four workers and four jobs, and the profit matrix is

$$P = \begin{pmatrix} 3 & 5 & 2 & 4 \\ 4 & 3 & 3 & 5 \\ 5 & 2 & 4 & 3 \\ 3 & 4 & 3 & 5 \end{pmatrix}.$$

By using the Kuhn-Munkres algorithm, we obtain the optimal assignment plan, which is shown in Table 5.3. The total profit is $f(P) = 5+3+5+5 = 18$.

Table 5.3: An optimal assignment plan

Worker	→	Job	Profit
1	→	2	5
2	→	3	3
3	→	1	5
4	→	4	5

In an uncertain environment, the profits p_{ij} are assumed to be nonnegative uncertain variables, and are rewritten as ξ_{ij}, respectively. The uncertain profit matrix is denoted as $\widetilde{P}$. The optimal assignment profit is a function of $\widetilde{P}$, which is denoted as $f(\widetilde{P})$. Obviously, the optimal assignment profit is also an uncertain variable. Denote $\Psi(x)$ as the uncertainty distribution of $f(\widetilde{P})$. For a given $\widetilde{P}$, how can we get $\Psi(x)$? The following subsection will give this answer.

Uncertainty Distribution $\Psi(x)$

For a deterministic profit matrix P, the optimal assignment profit $f(P)$ is strictly increasing with respect to P. That is,

$$f(P') \geq f(P'')$$

where $P' = (p'_{ij})_{n\times n}$, $P'' = (p''_{ij})_{n\times n}$, and $p'_{ij} \geq p''_{ij}$; and

$$f(P') > f(P'')$$

where $P' = (p'_{ij})_{n\times n}$, $P'' = (p''_{ij})_{n\times n}$, and $p'_{ij} > p''_{ij}$. According to Theorem 2.4, for a given $\widetilde{P}$, we can easily obtain the inverse uncertainty distribution of $f(\widetilde{P})$.

Theorem 5.9 *Let $\widetilde{P}$ be an uncertain profit matrix. If ξ_{ij} are independent uncertain variables with regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \ldots, n$, respectively, then the inverse uncertainty distribution of $f(\widetilde{P})$ is determined by*

$$\Psi^{-1}(\alpha) = f(P|p_{ij} = \Phi_{ij}^{-1}(\alpha)).$$

According to Theorem 5.9, we can obtain the uncertainty distribution of $f(\widetilde{P})$ in a numerical sense.

Example 5.4: Assume there are four workers for four jobs in a certain company, and the uncertain profit matrix is

$$\widetilde{P} = \begin{pmatrix} \mathcal{L}(2,5) & \mathcal{L}(1,3) & \mathcal{L}(2,4) & \mathcal{L}(3,6) \\ \mathcal{L}(3,4) & \mathcal{L}(3,5) & \mathcal{L}(3,6) & \mathcal{L}(1,5) \\ \mathcal{L}(1,3) & \mathcal{L}(2,6) & \mathcal{L}(1,4) & \mathcal{L}(3,4) \\ \mathcal{L}(2,4) & \mathcal{L}(2,3) & \mathcal{L}(2,5) & \mathcal{L}(2,3) \end{pmatrix}.$$

When $\alpha = 0.1$, we get a deterministic profit matrix $P = (p_{ij})_{4\times 4}$,

$$P = \begin{pmatrix} 2.3 & 1.2 & 2.2 & 3.3 \\ 3.1 & 3.2 & 3.3 & 1.4 \\ 1.2 & 2.4 & 1.3 & 3.1 \\ 2.2 & 2.1 & 2.3 & 2.1 \end{pmatrix}$$

where $p_{ij} = \Phi_{ij}^{-1}(0.1)$. Employing the Kuhn-Munkres algorithm, we have

$$\Psi^{-1}(0.1) = f(P|p_{ij} = \Phi_{ij}^{-1}(\alpha)) = 11.2,$$

that is, $\mathcal{M}\{f(\widetilde{P}) \leq 11.2\} = 0.1$.

When $\alpha = 0.9$, we get a deterministic profit matrix $P = (p_{ij})_{4\times 4}$,

$$P = \begin{pmatrix} 4.7 & 2.8 & 3.8 & 5.7 \\ 3.9 & 4.8 & 5.7 & 4.6 \\ 2.8 & 5.6 & 3.7 & 3.9 \\ 3.8 & 2.9 & 4.7 & 2.9 \end{pmatrix}$$

where $p_{ij} = \Phi_{ij}^{-1}(0.9)$. Employing the Kuhn-Munkres algorithm, we have

$$\Psi^{-1}(0.9) = f(P|p_{ij} = \Phi_{ij}^{-1}(\alpha)) = 13.3,$$

that is, $\mathcal{M}\{f(\widetilde{P}) \leq 13.3\} = 0.9$.

Decision Criteria

In an uncertain environment, some additional criteria should be employed to obtain the optimal assignment.

Given an uncertain profit matrix $\widetilde{P} = (p_{ij})_{n\times n}$, we use $\boldsymbol{x} = \{x_{ij}\}$ to denote a feasible assignment. Clearly, $\boldsymbol{x}$ is a feasible assignment if and only if

$$\begin{cases} \displaystyle\sum_{1\leq j\leq n} x_{ij} = 1 & \text{for } i = 1, 2, \ldots, n \\ \displaystyle\sum_{1\leq i\leq n} x_{ij} = 1 & \text{for } j = 1, 2, \ldots, n \\ x_{ij} = \{0, 1\} & \text{for } i, j = 1, 2, \ldots, n \end{cases}$$

where x_{ij} are decision variables, $i, j = 1, 2, \ldots, n$ respectively; that is, when $x_{ij} = 1$, worker i is assigned to job j; when $x_{ij} = 0$, worker i is not assigned to job j; the first constraint requires that every worker is assigned to exactly one job, and the second constraint requires that every job is assigned exactly one worker.

Thus, the profit of $\boldsymbol{x}$ is

$$w(\boldsymbol{x}) = \sum_{1\leq i,j\leq n} x_{ij}\xi_{ij}.$$

Unlike that in deterministic profit matrix P, the assignment profit is an uncertain variable in uncertain profit matrix $\widetilde{P}$. Then, how can we get the optimal assignment?

In fact, different criteria will result in different optimal assignments. In the following, we give three definitions of an optimal assignment from different criteria.

Definition 5.4 *Let $\widetilde{P}$ be an uncertain profit matrix, $\boldsymbol{x}^*$ a feasible assignment. Then $\boldsymbol{x}^*$ is called expected optimal assignment (EOA) if*

$$E[w(\boldsymbol{x}^*)] \geq E[w(\boldsymbol{x})]$$

holds for any feasible assignment $\boldsymbol{x}$.

Definition 5.5 *(Zhang-Peng [75]) Let $\widetilde{P}$ be an uncertain profit matrix, $\boldsymbol{x}^*$ a feasible assignment. Then $\boldsymbol{x}^*$ is called α-optimal assignment (α-OA) if*

$$\max\{W|\mathcal{M}\{w(\boldsymbol{x}^*) \geq W\} \geq \alpha\} \geq \max\{W|\mathcal{M}\{w(\boldsymbol{x}) \geq W\} \geq \alpha\}$$

holds for any feasible assignment $\boldsymbol{x}$, where α is a predetermined confidence level.

The definition of α-OA shows that, given $\alpha \in (0, 1)$, we hope to get a largest weight W such that uncertain variable $w(\boldsymbol{x}^*)$ is larger than W with confidence level α.

Let W be a satisfying predetermined minimal profit, the best optimal assignment of the uncertain assignment problem arises when the decision maker hopes that the uncertain measure of the profit greater then or equal to W will be as maximal as possible.

Definition 5.6 *Let $\widetilde{P}$ be an uncertain profit matrix, $\boldsymbol{x}^*$ a feasible assignment. Then $\boldsymbol{x}^*$ is called best optimal assignment (BOA) if*

$$\mathcal{M}\{w(\boldsymbol{x}^*) \geq W\} \geq \mathcal{M}\{w(\boldsymbol{x}) \geq W\}$$

holds for any feasible assignment $\boldsymbol{x}$, where W is a predetermined minimal profit.

Expected Optimal Assignment

In this section, we will give a method to find the expected optimal assignment. Firstly, based on the linearity of expectation, the following result is obtained.

Theorem 5.10 *Let $\widetilde{P} = (\xi_{ij})_{n\times n}$ be an uncertain profit matrix. If ξ_{ij} are independent uncertain variables, then the EOA of $\widetilde{P}$ is just the optimal assignment of P, where $P = (p_{ij})_{n\times n}$ is a profit matrix, and $p_{ij} = E[\xi_{ij}]$, $i, j = 1, 2, \ldots, n$, respectively.*

Proof: According to Definition 5.4, an expected value model for finding the EOR can be presented as follows,

$$\begin{cases} \max & E\left[\displaystyle\sum_{0\leq i,j\leq n} x_{ij}\xi_{ij}\right] & \\ s.t. & \displaystyle\sum_{1\leq j\leq n} x_{ij} = 1 & \text{for } i = 1, 2, \ldots, n \\ & \displaystyle\sum_{1\leq i\leq n} x_{ij} = 1 & \text{for } j = 1, 2, \ldots, n \\ & x_{ij} = \{0, 1\} & \text{for } i, j = 1, 2, \ldots, n. \end{cases} \tag{5.11}$$

Since ξ_{ij} are independent, according to Theorem 2.12, model (5.11) can be equivalently transformed to the following deterministic model:

$$\begin{cases} \max & \displaystyle\sum_{0\le i,j\le n} x_{ij}E[\xi_{ij}] & \\ s.t. & \displaystyle\sum_{1\le j\le n} x_{ij} = 1 & \text{for } i = 1,2,\ldots,n \\ & \displaystyle\sum_{1\le i\le n} x_{ij} = 1 & \text{for } j = 1,2,\ldots,n \\ & x_{ij} = \{0,1\} & \text{for } i,j = 1,2,\ldots,n. \end{cases} \tag{5.12}$$

In fact, model (5.12) describes the optimal assignment in profit matrix P, whose elements p_{ij} are $E[\xi_{ij}]$, $i,j = 1,2,\ldots,n$, respectively. The theorem is proved.

Theorem 5.10 provides an effect method to obtain EOA in uncertain profit matrix $\widetilde{P}$. Roughly speaking, the method to obtain EOA in uncertain profit matrix $\widetilde{P}$ can be summarized as follows,

Algorithm 5.4 (Algorithm for *EOA*)

Step 1. Calculate $E[\xi_{ij}]$, for each element ξ_{ij} in $\widetilde{P}$.

Step 2. Construct a deterministic profit matrix P, whose elements $p_{ij} = E[\xi_{ij}]$.

Step 3. Employ the Kuhn-Munkres algorithm to get the optimal assignment in profit matrix P.

The optimal assignment we get in Step 3 is just the EOA in uncertain profit matrix $\widetilde{P}$.

α-optimal Assignment

For confidence level α, the following theorem gives a method to obtain the α-OA.

Theorem 5.11 *(Zhang-Peng [75]) Let $\widetilde{P} = (\xi_{ij})_{n\times n}$ be an uncertain profit matrix. If ξ_{ij} are independent uncertain variables with regular uncertainty distributions Φ_{ij}, $i,j = 1,2,\ldots,n$, respectively. Then, the α-OA of $\widetilde{P}$ is just the optimal assignment of P, where $P = (p_{ij})_{n\times n}$ is a profit matrix, and $p_{ij} = \Phi_{ij}^{-1}(1-\alpha)$, $i,j = 1,2,\ldots,n$, respectively.*

Proof: According to Definition 5.5, a chance-constrained programming model

can be built to find the α-OA:

$$\begin{cases} \max & W \\ s.t. & \mathcal{M}\left\{\sum\limits_{1\le i,j\le n} x_{ij}\xi_{ij} \ge W\right\} \ge \alpha, \\ & \sum\limits_{1\le j\le n} x_{ij} = 1 & \text{for } i = 1,2,\ldots,n \\ & \sum\limits_{1\le i\le n} x_{ij} = 1 & \text{for } j = 1,2,\ldots,n \\ & x_{ij} = \{0,1\} & \text{for } i,j = 1,2,\ldots,n. \end{cases} \tag{5.13}$$

Since ξ_{ij} have regular uncertainty distributions Φ_{ij}, $i,j = 1,2,\ldots,n$, respectively. Then, according to Theorem 2.4, model (5.13) can be equivalently transformed to the following deterministic model:

$$\begin{cases} \max & W \\ s.t. & \sum\limits_{1\le i,j\le n} x_{ij}\Phi_{ij}^{-1}(1-\alpha) \ge W, \\ & \sum\limits_{1\le j\le n} x_{ij} = 1 & \text{for } i = 1,2,\ldots,n \\ & \sum\limits_{1\le i\le n} x_{ij} = 1 & \text{for } j = 1,2,\ldots,n \\ & x_{ij} = \{0,1\} & \text{for } i,j = 1,2,\ldots,n. \end{cases} \tag{5.14}$$

In more detail, model (5.14) is equivalent to

$$\begin{cases} \max & \sum\limits_{1\le i,j\le n} x_{ij}\Phi_{ij}^{-1}(1-\alpha) \\ s.t. & \sum\limits_{1\le j\le n} x_{ij} = 1 & \text{for } i = 1,2,\ldots,n \\ & \sum\limits_{1\le i\le n} x_{ij} = 1 & \text{for } j = 1,2,\ldots,n \\ & x_{ij} = \{0,1\} & \text{for } i,j = 1,2,\ldots,n. \end{cases} \tag{5.15}$$

In fact, the optimal solution to model (5.15) describes the optimal assignment in profit matrix P, whose elements p_{ij} are $\Phi_{ij}^{-1}(1-\alpha)$, $i,j = 1,2,\ldots,n$, respectively. Hence the theorem is proved.

As Theorem 5.11 illustrates, to obtain the α-OA, we only need to employ the Kuhn-Munkres algorithm to find the optimal assignment of a corresponding deterministic profit matrix P. The method for obtaining the α-OA is summarized in more detail in the following algorithm.

Algorithm 5.5 (Algorithm for α-OA)

Step 1. Calculate $\Phi_{ij}^{-1}(1-\alpha)$, for each element ξ_{ij} in $\widetilde{P}$.

Step 2. Construct a deterministic profit matrix P, whose elements $p_{ij} = \Phi_{ij}^{-1}(1-\alpha)$.

Step 3. Employ the Kuhn-Munkres algorithm to get the optimal assignment in profit matrix P.

The optimal assignment we get in Step 3 is just the α-OA in uncertain profit matrix $\widetilde{P}$.

Best Optimal Assignment

For the best optimal assignment, given a minimal profit W, we also prove that it can be obtained by a deterministic assignment matrix.

Theorem 5.12 *Let $\widetilde{P} = (\xi_{ij})_{n\times n}$ be an uncertain profit matrix. If ξ_{ij} are independent uncertain variables with regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \ldots, n$, respectively. Given a minimal profit W, the BOA of $\widetilde{P}$ is just the α-OA of $\widetilde{P}$, where $\Psi(W) = 1-\alpha$.*

Proof: According to Definition 5.6, the BOA is the optimal solution to the following uncertain programming model:

$$\begin{cases} \max & \mathcal{M}\left\{\sum_{1\le i,j\le n} x_{ij}\xi_{ij} \ge W\right\} & \\ s.t. & \sum_{1\le j\le n} x_{ij} = 1 & \text{for } i = 1, 2, \ldots, n \\ & \sum_{1\le i\le n} x_{ij} = 1 & \text{for } j = 1, 2, \ldots, n \\ & x_{ij} = \{0, 1\} & \text{for } i, j = 1, 2, \ldots, n. \end{cases} \tag{5.16}$$

Let

$$\mathcal{M}\left\{\sum_{1\le i,j\le n} x_{ij}\xi_{ij} \ge W\right\} \ge \alpha.$$

Then model (5.16) is equivalent to the following model:

$$\begin{cases} \max & \alpha \\ s.t. & \mathcal{M}\left\{\sum\limits_{1\le i,j\le n} x_{ij}\xi_{ij} \ge W\right\} \ge \alpha \\ & \sum\limits_{1\le j\le n} x_{ij} = 1 & \text{for } i = 1,2,\ldots,n \\ & \sum\limits_{1\le i\le n} x_{ij} = 1 & \text{for } j = 1,2,\ldots,n \\ & x_{ij} = \{0,1\} & \text{for } i,j = 1,2,\ldots,n. \end{cases} \tag{5.17}$$

As the proof of Theorem 5.11, the first constraint of model (5.17) can be reformulated as

$$\sum_{1\le i,j\le n} x_{ij}\Phi_{ij}^{-1}(1-\alpha) \ge W.$$

Thus, model (5.17) can be rebuilt as

$$\begin{cases} \max & \alpha \\ s.t. & \sum\limits_{1\le i,j\le n} x_{ij}\Phi_{ij}^{-1}(1-\alpha) \ge W \\ & \sum\limits_{1\le j\le n} x_{ij} = 1 & \text{for } i = 1,2,\ldots,n \\ & \sum\limits_{1\le i\le n} x_{ij} = 1 & \text{for } j = 1,2,\ldots,n \\ & x_{ij} = \{0,1\} & \text{for } i,j = 1,2,\ldots,n. \end{cases} \tag{5.18}$$

Set $\Psi(W) = 1-\overline{\alpha}$. We have $\Psi^{-1}(1-\overline{\alpha}) = W$ since Ψ is a regular uncertainty distribution. For the confidence level $\overline{\alpha}$, according to Theorem 5.11, we can get the $\overline{\alpha}$-OA plan $\overline{\boldsymbol{x}}$ for the uncertain profit matrix $\widetilde{P}$. In other words, $\overline{\boldsymbol{x}}$ is the optimal solution of model (5.14). That is, $\overline{\boldsymbol{x}}$ and $\overline{\alpha}$ satisfy the following constraints

$$\begin{cases} \sum\limits_{1\le i,j\le n} x_{ij}\Phi_{ij}^{-1}(1-\alpha) \ge W \\ \sum\limits_{1\le j\le n} x_{ij} = 1 & \text{for } i = 1,2,\ldots,n \\ \sum\limits_{1\le i\le n} x_{ij} = 1 & \text{for } j = 1,2,\ldots,n \\ x_{ij} = \{0,1\} & \text{for } i,j = 1,2,\ldots,n, \end{cases}$$

which means that $(\overline{\boldsymbol{x}}, \overline{\alpha})$ is a feasible solution to model (5.18), and $\overline{\alpha}$ is the objective value. In fact, we can prove that $\overline{\alpha}$ is also the optimal objective value of model (5.18).

Assume that $(\widetilde{\boldsymbol{x}}, \widetilde{\alpha})$ is another feasible solution of model (5.18), and $\tilde{\alpha}$ is the corresponding objective value such that $\widetilde{\alpha} > \overline{\alpha}$. Since Ψ is a regular uncertainty distribution, then

$$\Psi^{-1}(1-\widetilde{\alpha}) < \Psi^{-1}(1-\overline{\alpha}).$$

According to Theorem 5.9, we have

$$f(P|p_{ij} = \Phi_{ij}^{-1}(1-\widetilde{\alpha})) < f(P|p_{ij} = \Phi_{ij}^{-1}(1-\overline{\alpha})) = W.$$

It means that, for any feasible assignment $\boldsymbol{x} = \{x_{ij}\}$,

$$\sum_{1\leq i,j\leq n} x_{ij}\Phi_{ij}^{-1}(1-\widetilde{\alpha}) \leq f(P|p_{ij} = \Phi_{ij}^{-1}(1-\widetilde{\alpha})) < W, \tag{5.19}$$

since $f(P|p_{ij} = \Phi_{ij}^{-1}(1-\widetilde{\alpha}))$ is the optimal assignment profit of the profit matrix $P = (p_{ij})_{n\times n}$.

Obviously, expression (5.19) leads to a contradiction with the constraints in model (5.18). That is, the assumption is wrong.

It follows that $\overline{\alpha}$ is the optimal objective value, and the $\overline{\boldsymbol{x}}$ is the optimal solution of model (5.18). The theorem is proved.

For an uncertain profit matrix $\widetilde{P}$, given a budget profit level W, we present the following algorithm for finding the BOA.

Algorithm 5.6 (Algorithm for BOA)

Step 1. Obtain the uncertainty distribution function $\Psi(x)$ by interpolation method.

Step 2. Set $\Psi(W) = 1 - \alpha$.

Step 3. Employ the Kuhn-Munkres algorithm to get the α-OA for uncertain profit matrix $\widetilde{P}$.

As Theorem 5.12 illustrates, the α-OA we get in Step 3 is just the BOA we want, and the α is the corresponding optimal objective value.

Numerical Experiments

To illustrate the efficiency of the models and algorithms, we shall present some numerical experiments in the following context. We consider the assignment

problem on the following uncertain profit matrix

$$\widetilde{P} = \begin{pmatrix} \mathcal{Z}(2,5,7) & \mathcal{Z}(5,8,12) & \mathcal{Z}(4,8,9) & \mathcal{Z}(3,6,18) & \mathcal{Z}(4,8,10) \\ \mathcal{Z}(5,7,10) & \mathcal{Z}(3,5,8) & \mathcal{Z}(3,7,10) & \mathcal{Z}(2,4,7) & \mathcal{Z}(3,5,8) \\ \mathcal{Z}(4,7,9) & \mathcal{Z}(5,9,12) & \mathcal{Z}(3,5,8) & \mathcal{Z}(4,6,7) & \mathcal{Z}(3,6,8) \\ \mathcal{Z}(5,7,10) & \mathcal{Z}(3,5,9) & \mathcal{Z}(6,8,12) & \mathcal{Z}(2,5,8) & \mathcal{Z}(4,6,10) \\ \mathcal{Z}(4,5,9) & \mathcal{Z}(2,5,6) & \mathcal{Z}(3,4,7) & \mathcal{Z}(2,7,8) & \mathcal{Z}(3,5,17) \end{pmatrix}.$$

To obtain the expected optimal assignment plan, we first calculate the expected value $E[\xi_{ij}]$ for each ξ_{ij}, and then obtain a deterministic profit matrix as follows,

$$P = \begin{pmatrix} 4.75 & 8.25 & 7.25 & 8.25 & 7.5 \\ 7.25 & 5.25 & 6.75 & 4.25 & 5.25 \\ 6.75 & 8.75 & 5.25 & 5.75 & 5.75 \\ 7.25 & 5.5 & 8.5 & 5 & 6.5 \\ 5.75 & 4.5 & 4.5 & 6 & 7.5 \end{pmatrix}.$$

Following Algorithm 5.4, we find out the expected optimal assignment $\boldsymbol{x}$ which is presented in Table 5.4. The total profit of workers is $8.25 + 7.25 + 8.75 + 8.5 + 7.5 = 40.25$.

Table 5.4: The expected optimal assignment $\boldsymbol{x}$

Worker	$\rightarrow$	Job	Profit
1	$\rightarrow$	4	8.25
2	$\rightarrow$	1	7.25
3	$\rightarrow$	2	8.75
4	$\rightarrow$	3	8.5
5	$\rightarrow$	5	7.5

Let $\alpha = 0.8$. According to Algorithm 5.5, we first calculate the inverse uncertainty distribution $\Phi_{ij}^{-1}(0.2)$ for each profit ξ_{ij}, $i, j = 1, 2, \ldots, 5$, respectively, then we construct a deterministic profit matrix as follows,

$$P = \begin{pmatrix} 3.2 & 6.2 & 5.6 & 4.2 & 5.6 \\ 5.8 & 3.8 & 4.6 & 2.8 & 3.8 \\ 5.2 & 6.6 & 3.8 & 4.8 & 4.2 \\ 5.8 & 3.8 & 6.8 & 3.2 & 4.8 \\ 4.4 & 3.2 & 3.4 & 4 & 3.8 \end{pmatrix}.$$

It follows from Kuhn-Munkres algorithm and Theorem 5.11 that the 0.8-optimal assignment can be obtained, which is shown in Table 5.5, whose

Table 5.5: The 0.8-optimal assignment $\boldsymbol{x}$

Worker	$\rightarrow$	Job	Profit
1	$\rightarrow$	5	5.6
2	$\rightarrow$	1	5.8
3	$\rightarrow$	2	6.6
4	$\rightarrow$	3	6.8
5	$\rightarrow$	4	4

Table 5.6: List of $\Psi^{-1}(\alpha)$

α	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\Psi^{-1}(\alpha)$	25.4	28.8	32.2	35.6	39	41.8	48.6	55.4	62.2

total profit of workers is $5.6 + 5.8 + 6.6 + 6.8 + 4 = 28.8$. That is, 28.8 is the maximum profit that we could obtain with the confidence level $\alpha = 0.8$.

For different α, we can obtain the inverse uncertainty distribution $\Psi^{-1}(\alpha)$ of $f(\widetilde{P})$, which is listed in Table 5.6. According to Theorem 5.9, the uncertainty distribution of $f(\widetilde{P})$ can be obtained in a numerical sense, which is drawn by MATLAB in Figure 5.6.

Given $W = 28.8$, we have $\Psi(28.8) = 0.2 = 1 - 0.8$ from Table 5.6. Then, Theorem 5.18 illustrates that the best optimal assignment plan is just 0.8-optimal assignment, which is shown in Table 5.5.

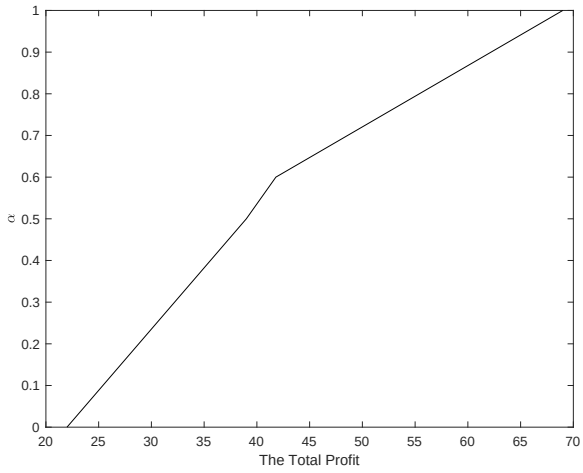


Figure 5.6: Uncertainty distribution of $f(\widetilde{P})$

5.3 Uncertain Shortest Path Problem

The shortest path problem is a special type of a combinatorial optimization problem that has extensive research in the last century. The shortest path problem can be briefly stated as: finding a path with minimum distance (time or cost) from the source node to the destination node. In the 1950s and 1960s, various approaches have been developed by Bellman [2], Dijkstra [13], and Floyd [17].

Generally, a network with edge weights is denoted as $N = (V, A, w)$, where $V = \{1, 2, \ldots, n\}$ is a finite set of vertices, $A = \{(i, j)|i, j \in V\}$ is the set of edges, and $w = \{w_{ij}|(i, j) \in A\}$ is the set of edge lengths. In a deterministic network, the edge lengths w_{ij} are crisp values, the shortest path length is a function of w, which is denoted as $f_{SP}(w)$. In other words, for a given w, $f_{SP}(w)$ can be obtained by using some effective algorithms, such as Dijkstra's algorithm.

Example 5.5: Let us consider the shortest path problem on the network $N = (V, A, w)$, which is shown in Figure 5.7.

When $w = \{w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}\} = \{2, 3, 6, 4, 3, 2\}$, the shortest path from vertex 1 to vertex 4 is $1 \to 2 \to 4$, whose total length is 5; that is, $f_{SP}(w) = w_{12} + w_{24} = 5$. When $w = \{w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}\} = \{1, 3, 5, 2, 4, 1\}$, the shortest path from vertex 1 to vertex 4 is $1 \to 2 \to 3 \to 4$, whose total length is 4; that is, $f_{SP}(w) = w_{12} + w_{23} + w_{34} = 4$.

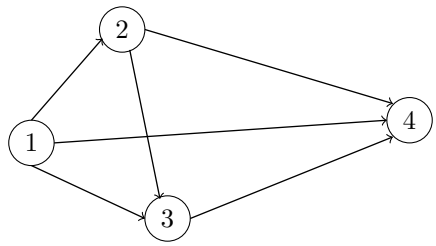


Figure 5.7: Network $N = (V, A, w)$

In an uncertain environment, Gao [23] first study the uncertain shortest path problem. Denote the network with uncertain edge lengths as $N = (V, A, \xi)$, where $\xi = \{\xi_{ij}|(i, j) \in A\}$ is the set of uncertain edge lengths. Thus, the shortest path length is a function of ξ, which is denoted as $f_{SP}(\xi)$. Obviously, $f_{SP}(\xi)$ is an uncertain variable. Denote the uncertainty distribution of $f_{SP}(\xi)$ as $\Psi(x)$. How can we find the distribution function $\Psi(x)$? The following section discusses this problem.

Uncertainty Distribution $\Psi(x)$

For a deterministic network $N = (V, A, w)$, the shortest path length is strictly increasing with respect to w. That is,

$$f_{SP}(w') \geq f_{SP}(w'')$$

where $w' = \{w'_{ij}|(i,j) \in A\}$, $w'' = \{w''_{ij}|(i,j) \in A\}$, and $w'_{ij} \geq w''_{ij}$; and

$$f_{SP}(w') > f_{SP}(w'')$$

where $w' = \{w'_{ij}|(i,j) \in A\}$, $w'' = \{w''_{ij}|(i,j) \in A\}$, and $w'_{ij} > w''_{ij}$.

According to Theorem 2.4, for a given w, we can easily obtain the inverse uncertainty distribution of $f_{SP}(\xi)$.

Theorem 5.13 *(Gao [23]) Let $N = (V, A, \xi)$ be an uncertain network with uncertain edge lengths ξ_{ij}. If ξ_{ij} are independent uncertain variables with regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \ldots, n$, respectively, then the inverse uncertainty distribution of $f_{SP}(\xi)$ can be obtained as follows,*

$$\Psi^{-1}(\alpha) = f_{SP}\left(w|w_{ij} = \Phi_{ij}^{-1}(\alpha)\right).$$

Example 5.6: Let us consider the uncertain shortest problem on an uncertain network $N = (V, A, \xi)$, whose topological graph structure is isomorphic with Figure 5.7. Suppose that the edge lengths are independent zigzag uncertain variables, which are shown as follows,

$$\xi_{12} \sim \mathcal{Z}(2,4,5), \qquad \xi_{13} \sim \mathcal{Z}(1,4,6), \qquad \xi_{14} \sim \mathcal{Z}(3,7,10),$$

$$\xi_{23} \sim \mathcal{Z}(2,3,5), \qquad \xi_{24} \sim \mathcal{Z}(1,3,4), \qquad \xi_{34} \sim \mathcal{Z}(2,4,5).$$

When $\alpha = 0.3$, we get the inverse uncertainty distribution $\Phi_{ij}^{-1}(0.3)$ of ξ_{ij} as follows,

$$\Phi_{12}^{-1}(0.3) = 3.2, \qquad \Phi_{13}^{-1}(0.3) = 2.8, \qquad \Phi_{14}^{-1}(0.3) = 5.4,$$

$$\Phi_{23}^{-1}(0.3) = 2.6, \qquad \Phi_{24}^{-1}(0.3) = 2.2, \qquad \Phi_{34}^{-1}(0.3) = 3.2.$$

Then, Dijkstra's algorithm tells us that the shortest path from vertex 1 to vertex 4 is $1 \to 4$, whose total length is 5.4. That is,

$$\Psi^{-1}(0.3) = f_{SP}\left(w|w_{ij} = \Phi_{ij}^{-1}(0.3)\right) = 5.4.$$

When $\alpha = 0.6$, we get the inverse uncertainty distribution $\Phi_{ij}^{-1}(0.6)$ of ξ_{ij} as follows,

$$\Phi_{12}^{-1}(0.6) = 4.2, \qquad \Phi_{13}^{-1}(0.6) = 4.4, \qquad \Phi_{14}^{-1}(0.6) = 7.6,$$

$$\Phi_{23}^{-1}(0.6) = 3.4, \qquad \Phi_{24}^{-1}(0.6) = 3.2, \qquad \Phi_{34}^{-1}(0.6) = 4.2.$$

Then, Dijkstra's algorithm tells us that the shortest path from vertex 1 to vertex 4 is $1 \to 2 \to 4$, whose total length is 7.4. That is,

$$\Psi^{-1}(0.6) = f_{SP}\left(w | w_{ij} = \Phi_{ij}^{-1}(0.6)\right) = 7.4.$$

When $\alpha = 0.9$, we get the inverse uncertainty distribution $\Phi_{ij}^{-1}(0.9)$ of ξ_{ij} as follows,

$$\Phi_{12}^{-1}(0.9) = 4.8, \qquad \Phi_{13}^{-1}(0.9) = 5.6, \qquad \Phi_{14}^{-1}(0.9) = 9.4,$$
$$\Phi_{23}^{-1}(0.9) = 4.6, \qquad \Phi_{24}^{-1}(0.9) = 3.8, \qquad \Phi_{34}^{-1}(0.9) = 4.8.$$

Then, Dijkstra's algorithm tells us that the shortest path from vertex 1 to vertex 4 is $1 \to 2 \to 4$, whose total length is 8.6. That is,

$$\Psi^{-1}(0.9) = f_{SP}\left(w | w_{ij} = \Phi_{ij}^{-1}(0.9)\right) = 8.6.$$

Clearly, for any $\alpha \in (0, 1)$, we can find the uncertainty distribution function $\Psi(x)$ of $f_{SP}(\xi)$ by repeating this process.

Decision Criteria

In an uncertain network $N = (V, A, \xi)$, P is a path from vertex s to vertex t if and only if

$$\begin{cases} \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = s, \\ 0, & i \in V \setminus \{s, t\} \\ -1, & i = t, \end{cases} \\ x_{ij} = \{0, 1\} \quad \text{for } (i, j) \in A \end{cases}$$

where $x_{ij} \in \{0, 1\}$ is a decision variable; that is, when $x_{ij} = 1$, edge (i, j) is in path P; when $x_{ij} = 0$, edge (i, j) is not in path P. Then the length of P is

$$l(P) = \sum_{(i,j)\in A} x_{ij}\xi_{ij}.$$

Clearly, $l(P)$ is also an uncertain variable. For two uncertain variables, it is difficult for us to rank them directly. Thus, some decision criteria should be employed to obtain the optimal decision.

The first way adopts the expected value criterion to rank uncertain variables. The concept of the expected shortest path is defined in the following way.

Definition 5.7 *Let $N = (V, A, \xi)$ be an uncertain network, and P^* be a path from the source vertex to the destination vertex. Then P^* is called expected shortest path if*

$$E[l(\boldsymbol{x}^*)] \leq E[l(\boldsymbol{x})]$$

holds for any path P from the source vertex to the destination vertex.

Given an $\alpha \in (0,1)$, the decision maker hopes to get the smallest value L such that uncertain variable P^* is less than or equal to L with confidence level α. This produces the concept of α-shortest path.

Definition 5.8 *(Gao [23]) Let $N = (V, A, \xi)$ be an uncertain network, and P^* be a path from the source vertex to the destination vertex. Then P^* is called α-shortest path if*

$$\min\{L|\mathcal{M}\{l(P^*) \leq L\} \geq \alpha\} \leq \min\{L|\mathcal{M}\{l(P) \leq L\} \geq \alpha\}$$

holds for any path P from the source vertex to the destination vertex, where α is a predetermined confidence level.

Let L be a predetermined minimal length, such that the best shortest path of uncertain shortest path problem arises when the decision maker hopes that the uncertain measure of the uncertain length $l(P)$ less then or equal to L will be as maximal as possible.

Definition 5.9 *(Gao [23]) Let $N = (V, A, \xi)$ be an uncertain network, and P^* be a path from the source vertex to the destination vertex. Then P^* is called best shortest path if*

$$\mathcal{M}\{l(P^*) \leq L\} \geq \mathcal{M}\{l(P) \leq L\}$$

holds for any path P from the source vertex to the destination vertex, where L is a predetermined length.

Expected Shortest Path

Theorem 5.14 *Let $N = (V, A, \xi)$ be an uncertain network. If ξ_{ij} are independent uncertain variables, then the expected shortest path of $N = (V, A, \xi)$ is just the shortest path of $N = (V, A, w)$, where $w_{ij} = E[\xi_{ij}]$, $(i, j) \in A$.*

Proof: Assume that vertex s is the source vertex and vertex t is the destination vertex. According to Definition 5.7, an expected value model for finding the expected shortest path is constructed as follows,

$$\begin{cases} \min & E\left[\displaystyle\sum_{0\leq i,j\leq n} x_{ij}\xi_{ij}\right] \\ s.t. & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = s, \\ 0, & i \in V \setminus \{s,t\} \\ -1, & i = t, \end{cases} \\ & x_{ij} = \{0,1\} \quad \text{for } (i,j) \in A. \end{cases} \tag{5.20}$$

Since ξ_{ij} are independent uncertain variables, according to Theorem 2.12, model (5.20) can be equivalently transformed to the following deterministic model:

$$\begin{cases} \min & \displaystyle\sum_{0\le i,j\le n} x_{ij}E[\xi_{ij}] \\ s.t. & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i=s, \\ 0, & i\in V\setminus\{s,t\} \\ -1, & i=t, \end{cases} \\ & x_{ij}=\{0,1\} \quad \text{for } (i,j)\in A. \end{cases} \tag{5.21}$$

Obviously, the optimal solution of model (5.21) is the shortest path in a deterministic $N=(V,A,w)$, where $w_{ij}=E[\xi_{ij}]$. The theorem is proved.

According to Theorem 5.14, we can obtain the expected shortest path by the following algorithm.

Algorithm 5.7 (Algorithm for Expected Shortest Path)

Step 1. Calculate $E[\xi_{ij}]$ for each length ξ_{ij}.

Step 2. Construct a deterministic network $N=(V,A,w)$, where $w_{ij}=E[\xi_{ij}]$.

Step 3. Employ Dijkstra's algorithm to get the shortest path in network N.

The shortest path we get in Step 3 is just the expected shortest path in uncertain network $N=(V,A,\xi)$.

α-shortest Path

Theorem 5.15 *(Gao [23]) Let $N=(V,A,\xi)$ be an uncertain network, where ξ_{ij} are independent uncertain variables with regular uncertainty distributions Φ_{ij}, $(i,j)\in A$. Then, the α-shortest path of $N=(V,A,\xi)$ is just the shortest path of $N=(V,A,w)$, where $w_{ij}=\Phi_{ij}^{-1}(\alpha)$.*

Proof: Assume that vertex s is the source vertex and vertex t is the destination vertex. According to Definition 5.8, a chance-constrained programming model can be built to find the α-shortest path:

$$\begin{cases} \min & L \\ s.t. & \mathcal{M}\left\{\sum_{0\le i,j\le n} x_{ij}\xi_{ij} \le L\right\} \ge \alpha \\ & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = s, \\ 0, & i \in V \setminus \{s,t\} \\ -1, & i = t, \end{cases} \\ & x_{ij} = \{0,1\} \quad \text{for } (i,j) \in A. \end{cases} \tag{5.22}$$

Since ξ_{ij} have regular uncertainty distributions Φ_{ij}, according to Theorem 2.4, model (5.22) can be equivalently transformed to the following deterministic model:

$$\begin{cases} \min & T \\ s.t. & \sum_{1\le i,j\le n} x_{ij}\Phi_{ij}^{-1}(\alpha) \le L, \\ & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = s, \\ 0, & i \in V \setminus \{s,t\} \\ -1, & i = t, \end{cases} \\ & x_{ij} = \{0,1\} \quad \text{for } (i,j) \in A. \end{cases} \tag{5.23}$$

Obviously, model (5.23) is equivalent to

$$\begin{cases} \min & \sum_{1\le i,j\le n} x_{ij}\Phi_{ij}^{-1}(\alpha) \\ s.t. & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = s, \\ 0, & i \in V \setminus \{s,t\} \\ -1, & i = t, \end{cases} \\ & x_{ij} = \{0,1\} \quad \text{for } (i,j) \in A. \end{cases} \tag{5.24}$$

As we know, the optimal solution to model (5.24) is just the shortest path on network $N = (V, A, w)$, where $w_{ij} = \Phi^{-1}(\alpha)$. Hence the theorem is proved.

According to Theorem 5.15, for an uncertain network $N = (V, A, \xi)$, the following algorithm can be used to obtain the α-shortest path.

Algorithm 5.8 (Algorithm for α-shortest Path)

Step 1. Calculate $\Phi_{ij}^{-1}(\alpha)$ for each element ξ_{ij} in $N = (V, A, \xi)$.

Step 2. Construct a deterministic network $N = (V, A, w)$, where $w_{ij} = \Phi_{ij}^{-1}(\alpha)$.

Step 3. Employ Dijkstra's algorithm to get the shortest path in network N.

The optimal assignment we get in Step 3 is just the α-shortest path in uncertain network $N = (V, A, \xi)$.

Best Shortest Path

For an uncertain network $N = (V, A, \xi)$, assume that vertex s is the source and vertex t is the destination. According to Definition 5.9, a measure-chance programming is built to find the best shortest path,

$$\begin{cases} \max & \mathcal{M}\left\{\sum_{(i,j)\in A} x_{ij}\xi_{ij} \le L\right\} \\ s.t. & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = s, \\ 0, & i \in V \setminus \{s,t\} \\ -1, & i = t, \end{cases} \\ & x_{ij} = \{0,1\} \quad \text{for } (i,j) \in A. \end{cases} \tag{5.25}$$

The following theorem provides a good method to obtain the best shortest path in uncertain network $N = (V, A, \xi)$.

Theorem 5.16 *(Gao [23]) Let $N = (V, A, \xi)$ be an uncertain network, where ξ_{ij} are independent uncertain variables with regular uncertainty distributions Φ_{ij}, $(i,j) \in A$. Given a predetermined length L, the best shortest path is just the α-shortest path of $N = (V, A, \xi)$, where $\alpha = \Psi(L)$.*

Proof: As we know, the most shortest path is the optimal solution to the model (5.25). Set

$$\mathcal{M}\left\{\sum_{(i,j)\in A} x_{ij}\xi_{ij} \le L\right\} \ge \alpha,$$

then model (5.25) is equivalent to the following model:

$$\begin{cases} \max & \alpha \\ s.t. & \mathcal{M}\left\{\sum_{0\le i,j\le n} x_{ij}\xi_{ij} \le L\right\} \ge \alpha \\ & \sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = s, \\ 0, & i \in V \setminus \{s,t\} \\ -1, & i = t, \end{cases} \\ & x_{ij} = \{0,1\} \quad \text{for } (i,j) \in A. \end{cases} \tag{5.26}$$

As the proof of Theorem 5.15, the first constraint of model (5.26) can be reformulated as

$$\sum_{(i,j)\in A} x_{ij}\Phi_{ij}^{-1}(\alpha) \leq L.$$

That is, model (5.26) can be rebuilt as follows,

$$\begin{cases} \max & \alpha \\ s.t. & \displaystyle\sum_{(i,j)\in A} x_{ij}\Phi_{ij}^{-1}(\alpha) \leq L \\ & \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = s, \\ 0, & i \in V \setminus \{s,t\} \\ -1, & i = t, \end{cases} \\ & x_{ij} = \{0,1\} \quad \text{for } (i,j) \in A. \end{cases} \tag{5.27}$$

Set $\alpha' = \Psi(L)$. Since ξ_{ij} have regular uncertainty distributions, according to Theorem 2.4, Ψ is also a regular uncertainty distribution. Then, we have $\Psi^{-1}(\alpha') = L$. Let P' be the α'-shortest path. This means that P' is the optimal solution of model (5.23). Thus, P' and α' satisfy the following constraints

$$\begin{cases} \displaystyle\sum_{(i,j)\in A} x_{ij}\Phi_{ij}^{-1}(\alpha) \leq L \\ \displaystyle\sum_{j:(i,j)\in A} x_{ij} - \sum_{j:(j,i)\in A} x_{ji} = \begin{cases} 1, & i = s, \\ 0, & i \in V \setminus \{s,t\} \\ -1, & i = t, \end{cases} \\ x_{ij} = \{0,1\} \quad \text{for } (i,j) \in A. \end{cases}$$

This shows that P' is a feasible solution to model (5.27), and α' is the objective. In fact, α' is also the optimal objective.

Assume that $\overline{P}$ is a feasible solution of model (5.27) with the objective $\overline{\alpha}$ such that $\overline{\alpha} > \alpha'$. As we know, the inverse distribution Ψ^{-1} is strictly increasing since Ψ is a regular uncertainty distribution. That is,

$$\Psi^{-1}(\overline{\alpha}) > \Psi^{-1}(\alpha') = L.$$

Which means that,

$$f_{SP}\left(\Phi_{ij}^{-1}(\overline{\alpha})|(i,j) \in A\right) > f_{SP}\left(\Phi_{ij}^{-1}(\alpha')|(i,j) \in A\right) = L.$$

In more detail, for any path P from vertex s to destination vertex t,

$$\sum_{(i,j)\in P} x_{ij}\Phi_{ij}^{-1}(\overline{\alpha}) \geq f_{SP}(\Phi_{ij}^{-1}(\overline{\alpha})|(i,j) \in A) > L. \tag{5.28}$$

Obviously, inequality (5.28) leads to a contradiction with the constraints in model (5.27). It shows that $\alpha' = \Psi(L)$ is the optimal objective value, and then P' is the optimal solution of model (5.27). The theorem is proved.

In uncertain network $N = (V, A, \xi)$, given a predetermined length W, Theorem 5.16 provides a good method to obtain the best shortest path.

Algorithm 5.9 (Algorithm for best shortest path)

Step 1. Obtain the uncertainty distribution $\Psi(x)$ of $f_{SP}(\xi)$ by interpolation method.

Step 2. Set $\alpha = \Psi(L)$.

Step 3. Search the α-shortest path under the confidence level α.

According to Theorem 5.16, the α-shortest path we get in Step 3 is simply the best shortest path in $N = (V, A, \xi)$ we want.

Numerical Experiments

In this section, some numerical examples of the uncertain shortest path problem is presented to show the efficiency of the models and algorithms. We consider an uncertain network $N = (V, A, \xi)$ with 8 vertices and 14 edges, depicted in Figure 5.8. Assume that vertex 1 is the source and vertex 8 is the destination. The lengths of edges are independent zigzag uncertain variables ξ_{ij}, which are listed in Table 5.7. We also assume that $\xi_{ij} = \xi_{ji}$.

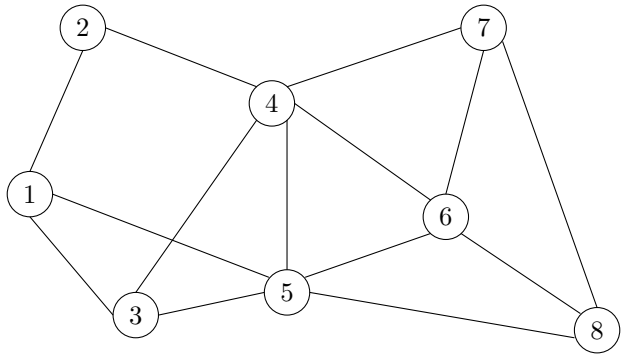


Figure 5.8: Uncertain network $N = (V, A, \xi)$

First, if we want to search an expected shortest path from vertex 1 to vertex 8, we can calculate expected value $E(\xi_{ij})$ for each length ξ_{ij}, $(i, j) \in A$. The values are listed in the second column of Table 5.8. Then we can construct a deterministic network $N = (V, A, w)$, where $w_{ij} = E[\xi_{ij}]$. Dijkstra's algorithm tells us that the shortest path is

$$1 \to 5 \to 8,$$

Table 5.7: List of edge lengths

edge (i,j)	ξ_{ij}	edge (i,j)	ξ_{ij}
(1,2)	$\mathcal{Z}(3,6,8)$	(4,6)	$\mathcal{Z}(5,6,8)$
(1,3)	$\mathcal{Z}(2,5,7)$	(4,7)	$\mathcal{Z}(4,7,11)$
(1,5)	$\mathcal{Z}(5,8,12)$	(5,6)	$\mathcal{Z}(4,5,7)$
(2,4)	$\mathcal{Z}(5,7,10)$	(5,8)	$\mathcal{Z}(6,10,12)$
(3,4)	$\mathcal{Z}(4,7,9)$	(6,7)	$\mathcal{Z}(4,6,7)$
(3,5)	$\mathcal{Z}(4,6,9)$	(6,8)	$\mathcal{Z}(2,4,5)$
(4,5)	$\mathcal{Z}(3,5,6)$	(7,8)	$\mathcal{Z}(5,8,10)$

whose total length is 17.5.

Table 5.8: The values of $E[\xi_{ij}]$, $\Phi_{ij}^{-1}(0.9)$ and $\Phi_{ij}^{-1}(0.95)$

edge (i,j)	$E[\xi_{ij}]$	$\Phi_{ij}^{-1}(0.9)$	$\Phi_{ij}^{-1}(0.95)$
(1,2)	5.75	7.6	7.8
(1,3)	4.75	6.6	6.8
(1,5)	8.25	11.2	11.6
(2,4)	7.25	9.4	9.7
(3,4)	6.75	8.6	8.8
(3,5)	6.25	8.4	8.7
(4,5)	4.75	5.8	5.9
(4,6)	6.25	7.6	7.8
(4,7)	7.25	10.2	10.6
(5,6)	5.25	6.6	6.8
(5,8)	9.25	11.6	11.8
(6,7)	5.75	6.8	6.9
(6,8)	4	4.8	4.9
(7,8)	7.75	9.6	9.8

Given $\alpha = 0.9$, we want to search the 0.9-shortest path. As Algorithm 5.8 indicates, we should first calculate $\Phi_{ij}^{-1}(0.9)$ for each ξ_{ij}, $(i,j) \in A$, which is shown in the third column of Table 5.8. Then we construct a deterministic network $N = (V, A, w)$, where $w_{ij} = \Phi_{ij}^{-1}(0.9)$, $(i,j) \in A$. The 0.9-shortest path is just the optimal plan in network $N = (V, A, w)$. Dijkstra's algorithm tells us that the optimal plan is

$$1 \to 5 \to 6 \to 8,$$

and the total length is 22.6.

Given $\alpha = 0.95$, the values of $\Phi_{ij}^{-1}(0.95)$ are listed in the fourth column

of Table 5.8. By the same method, we can obtain the 0.95-shortest path is

$$1 \to 5 \to 6 \to 8,$$

and the total length is 23.3. Choosing different α, we obtain Table 5.9.

Obviously, as the confidence level α increases, the total length will increase. It is because the greater α is, the more conservative the plan is, which results in greater length. Repeating this process, we obtain the uncertainty distribution of $f_{SP}(\xi)$ in a numerical sense, which is drawn by MATLAB in Figure 5.9.

Table 5.9: List of $\Psi^{-1}(\alpha)$

α	α-shortest path	$\Psi^{-1}(\alpha)$
0.1	$1 \to 5 \to 8$	11.6
0.2	$1 \to 5 \to 8$	13.2
0.3	$1 \to 5 \to 8$	14.8
0.4	$1 \to 5 \to 6 \to 8$	16
0.5	$1 \to 5 \to 6 \to 8$	17
0.6	$1 \to 5 \to 6 \to 8$	18.4
0.7	$1 \to 5 \to 6 \to 8$	19.8
0.8	$1 \to 5 \to 6 \to 8$	21.2
0.9	$1 \to 5 \to 6 \to 8$	22.6

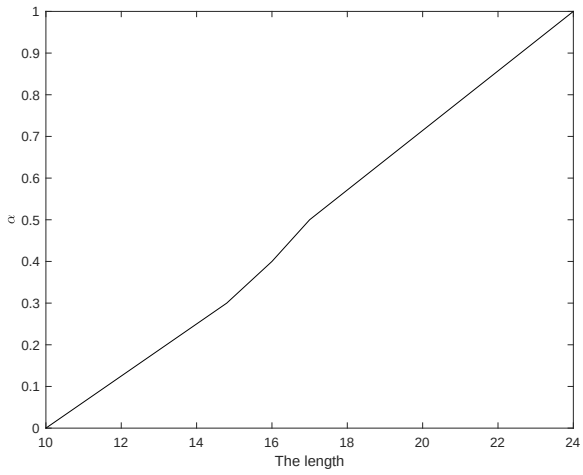


Figure 5.9: Uncertainty distribution of $f_{SP}(\xi)$

After obtaining the distribution function of $f_{SP}(\xi)$, we can further obtain

the best shortest path. Given a length $L = 16$, from Table 5.9, we have $\Psi(16) = 0.4$. Then, Theorem 5.16 tells us that the best shortest path we want to obtain is just the 0.4-shortest path, which is shown in Table 5.9.

Chapter 6

Applications of Uncertain Network Optimization

6.1 Transportation Problem

It is well-known that the transportation problem plays an important role in logistics and supply chain management for reducing cost and improving the service quality. The solid transportation problem was introduced by Haley [33], which was defined as transportation of goods by different conveyances. As another extension of the traditional transportation problem, the charge transportation problem was initialized by Hirsch-Dantzig [35].

Suppose that there are m sources, n destinations, and l conveyances. The solid transportation problem aims to make a transport plan so that the total transportation cost is minimized. In the fixed charge solid transportation problem, two types of costs will be taken into consideration, the direct cost and the fixed charge. The direct cost is the cost with respect to per unit transportation amount. The fixed charge will be paid when the transportation activity between a source and a destination by a conveyance occurs.

The fixed charge solid transportation problem in an uncertain environment was first studied by Zhang-Peng-Li-Chen [76]. In order to solve the problem by mathematical model, some notations and decision variables are listed in the following.

Notations

$i \in \{1, 2, \ldots, m\}$: The index for sources;

$j \in \{1, 2, \ldots, n\}$: The index for destinations;

$k \in \{1, 2, \ldots, l\}$: The index for conveyances;

$\tilde{a}_i$: The amount of products in source i which can be transported to n destinations;

$\tilde{b}_j$: The minimal demand of products in destination j;

$\tilde{c}_k$: The transportation capacity of conveyance k;

ξ_{ijk}: The direct cost of unit transportation amount from source i to destination j by conveyance k;

η_{ijk}: The fixed charge with respect to transportation activity from source i to destination j by conveyance k;

Decision variables

x_{ijk}: The quantity transported from source i to destination j by conveyance k.

$$y_{ijk} = \begin{cases} 1, & \text{if } x_{ijk} > 0 \\ 0, & \text{otherwise} \end{cases}$$

where $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, l$, respectively. This implies that, if the transportation activity is assigned from source i to destination j by conveyance k, then the corresponding fixed charge will occur.

In uncertain environments, the supplies $\tilde{a}_i$, the demands $\tilde{b}_j$, the conveyance capacities $\tilde{c}_k$, the direct costs ξ_{ijk} and the fixed charges η_{ijk} are assumed to be uncertain variables, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, l$, respectively.

Mathematical Models

Based on the notations defined as mentioned above, the total cost of the uncertain fixed charge solid transportation problem is

$$f(\boldsymbol{x}, \boldsymbol{y}; \xi, \eta) = \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{l} (\xi_{ijk} x_{ijk} + \eta_{ijk} y_{ijk})$$

where $\boldsymbol{x}, \boldsymbol{y}, \xi$ and η denote the vectors consisting of $x_{ijk}, y_{ijk}, \xi_{ijk}$, and η_{ijk}, $i = 1, 2, \ldots, m$; $j = 1, 2, \ldots, n$; $k = 1, 2, \ldots, l$, respectively.

It is clear that $f(\boldsymbol{x}, \boldsymbol{y}; \xi, \eta)$ is also an uncertain variable. To obtain the optimal transportation plan in an uncertain environment, we propose some decision criteria.

The first way adopts the expected value criterion to rank uncertain variables. The expected value is the average value of an uncertain variable in the sense of an uncertain measure.

Definition 6.1 *(Zhang-Peng-Li-Chen [76]) A solution* $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ *is called the expected transportation plan of the uncertain fixed charge solid transportation problem if*

$$E[f(\boldsymbol{x}^*, \boldsymbol{y}^*; \xi, \eta)] \leq E[f(\boldsymbol{x}, \boldsymbol{y}; \xi, \eta)]$$

holds for any feasible solution $(\boldsymbol{x}, \boldsymbol{y})$.

According to Definition 6.1, the expected value model for the uncertain fixed charge solid transportation problem was constructed by Zhang-Peng-Li-Chen [76]:

$$\begin{cases} \min & E\left[\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}(\xi_{ijk}x_{ijk}+\eta_{ijk}y_{ijk})\right] \\ s.t. & E\left[\sum_{j=1}^{n}\sum_{k=1}^{l}x_{ijk}-\tilde{a}_i\right]\leq 0, \quad i=1,2,\ldots,m \\ & E\left[\sum_{i=1}^{m}\sum_{k=1}^{l}x_{ijk}-\tilde{b}_j\right]\geq 0, \quad j=1,2,\ldots,n \\ & E\left[\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk}-\tilde{c}_k\right]\leq 0, \quad k=1,2,\ldots,l \\ & x_{ijk}\geq 0, \\ & y_{ijk}=\begin{cases} 1, & \text{if } x_{ijk}>0 \\ 0, & \text{otherwise,} \end{cases} \\ & i=1,2,\ldots,m; \quad j=1,2,\ldots,n; \quad k=1,2,\ldots,l. \end{cases} \tag{6.1}$$

In the sense of expected value, the first constraint of model (6.1) requires that the total amount transported from source i is no more than its capacity; the second constraint of model (6.1) requires that the total amount transported from m sources should satisfy the demand of destination j; the third constraint is a capacity constraint for conveyance k.

Under other conditions, given an $\alpha \in (0,1)$, the decision maker hopes to get a smallest value $\bar{f}$ such that uncertain cost $f(\boldsymbol{x}^*, \boldsymbol{y}^*; \xi, \eta)$ is less than $\bar{f}$ with confidence level α, which causes the appearance of the following criterion:

Definition 6.2 *(Zhang-Peng-Li-Chen [76]) A solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is said to be α-transportation plan of the uncertain fixed charge solid transportation problem if*

$$\min\{\bar{f}|\mathcal{M}\{f(\boldsymbol{x}^*, \boldsymbol{y}^*; \xi, \eta) \leq \bar{f}\} \geq \alpha\} \leq \min\{\bar{f}|\mathcal{M}\{f(\boldsymbol{x}, \boldsymbol{y}; \xi, \eta) \leq \bar{f}\} \geq \alpha\}$$

holds for any feasible solution $(\boldsymbol{x}, \boldsymbol{y})$, and $\alpha \in (0,1)$ is a predetermined confidence level.

Chance-constrained programming is another well-method to deal with the optimality problem in an uncertain environment. According to Definition 6.2, a chance-constrained programming model for the uncertain fixed charge solid transportation problem was constructed by Zhang-Peng-Li-Chen [76]:

$$
\begin{cases}
\min \quad \bar{f} \\
s.t. \quad \mathcal{M}\left\{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{l}(\xi_{ijk}x_{ijk}+\eta_{ijk}y_{ijk}) \le \bar{f}\right\} \ge \alpha \\
\qquad \mathcal{M}\left\{\sum\limits_{j=1}^{n}\sum\limits_{k=1}^{l}x_{ijk} \le \tilde{a}_i\right\} \ge \alpha_i, \quad i=1,2,\ldots,m \\
\qquad \mathcal{M}\left\{\sum\limits_{i=1}^{m}\sum\limits_{k=1}^{l}x_{ijk} \ge \tilde{b}_j\right\} \ge \beta_j, \quad j=1,2,\ldots,n \\
\qquad \mathcal{M}\left\{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}x_{ijk} \le \tilde{c}_k\right\} \ge \gamma_k, \quad k=1,2,\ldots,l \\
\qquad x_{ijk} \ge 0, \\
\qquad y_{ijk} = \begin{cases} 1, & \text{if } x_{ijk} > 0 \\ 0, & \text{otherwise,} \end{cases} \\
\qquad i=1,2,\ldots,m; \quad j=1,2,\ldots,n; \quad k=1,2,\ldots,l
\end{cases}
\tag{6.2}
$$

where α, $\alpha_i (i=1,2,\ldots,m)$, $\beta_j (j=1,2,\ldots,n)$ and $\gamma_k (k=1,2,\ldots,l)$ are predetermined confidence levels.

Given a predetermined confidence cost level $\bar{f}$, the best transportation plan of the uncertain fixed charge solid transportation problem arises when the decision maker hopes that the measure of the uncertain cost $f(\boldsymbol{x}^*, \boldsymbol{y}^*; \xi, \eta)$ less than or equal to $\bar{f}$ will be as maximal as possible.

Definition 6.3 *(Zhang-Peng-Li-Chen [76]) A solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is said to be the best transportation plan of the uncertain fixed charge solid transportation problem if*

$$\mathcal{M}\{f(\boldsymbol{x}^*, \boldsymbol{y}^*; \xi, \eta) \le \bar{f}\} \ge \mathcal{M}\{f(\boldsymbol{x}, \boldsymbol{y}; \xi, \eta) \le \bar{f}\}$$

holds for any feasible solution $(\boldsymbol{x}, \boldsymbol{y})$, and $\bar{f}$ is a predetermined confidence cost level.

If the modeling idea of the most transportation plan is approved, we may construct a measure-chance programming model under the chance constraints as follows (Zhang-Peng-Li-Chen [76]),

$$\begin{cases} \max & \mathcal{M}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}(\xi_{ijk}x_{ijk}+\eta_{ijk}y_{ijk}) \le \bar{f}\right\} \\ s.t. & \mathcal{M}\left\{\sum_{j=1}^{n}\sum_{k=1}^{l}x_{ijk} \le \tilde{a}_i\right\} \ge \alpha_i, \quad i=1,2,\ldots,m \\ & \mathcal{M}\left\{\sum_{i=1}^{m}\sum_{k=1}^{l}x_{ijk} \ge \tilde{b}_j\right\} \ge \beta_j, \quad j=1,2,\ldots,n \\ & \mathcal{M}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk} \le \tilde{c}_k\right\} \ge \gamma_k, \quad k=1,2,\ldots,l \\ & x_{ijk} \ge 0, \quad y_{ijk} = \begin{cases} 1, & \text{if } x_{ijk} > 0 \\ 0, & \text{otherwise,} \end{cases} \\ & i=1,2,\ldots,m; \quad j=1,2,\ldots,n; \quad k=1,2,\ldots,l \end{cases} \tag{6.3}$$

where $\alpha_i (i = 1,2,\ldots,m)$, $\beta_j (j = 1,2,\ldots,n)$ and $\gamma_k (k = 1,2,\ldots,l)$ are predetermined confidence levels.

Discussion of the Models

Note that there are many uncertain variables in the above models. To solve them, it is necessary for us to compute the expected value, critical value or uncertain measure of uncertain variables. In the following, we will discuss the properties of the proposed models.

Theorem 6.1 *(Zhang-Peng-Li-Chen [76]) If $\tilde{a}_i$, $\tilde{b}_j$, $\tilde{c}_k$, ξ_{ijk} and η_{ijk} are independent uncertain variables, then model (6.1) is equivalent to the following model*

$$\begin{cases} \min & \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}(x_{ijk}E[\xi_{ijk}]+y_{ijk}E[\eta_{ijk}]) \\ s.t. & \sum_{j=1}^{n}\sum_{k=1}^{l}x_{ijk} \le E[\tilde{a}_i], \quad i=1,2,\ldots,m \\ & \sum_{i=1}^{m}\sum_{k=1}^{l}x_{ijk} \ge E[\tilde{b}_j], \quad j=1,2,\ldots,n \\ & \sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk} \le E[\tilde{c}_k], \quad k=1,2,\ldots,l \\ & x_{ijk} \ge 0, \quad y_{ijk} = \begin{cases} 1, & \textit{if } x_{ijk} > 0 \\ 0, & \textit{otherwise,} \end{cases} \\ & i=1,2,\ldots,m; \quad j=1,2,\ldots,n; \quad k=1,2,\ldots,l. \end{cases} \tag{6.4}$$

Proof: By using the linearity of expectation operator of uncertain variables, we can easily prove this theorem.

Example 6.1: Suppose the uncertain parameters $\tilde{a}_i$, $\tilde{b}_j$, $\tilde{c}_k$, ξ_{ijk} and η_{ijk} are independent linear uncertain variables, i.e., $\tilde{a}_i \sim \mathcal{L}(a_{i_1}, a_{i_2})$, $\tilde{b}_j \sim \mathcal{L}(b_{j_1}, b_{j_2})$, $\tilde{c}_k \sim \mathcal{L}(c_{k_1}, c_{k_2})$, $\xi_{ijk} \sim \mathcal{L}(d_{(ijk)_1}, d_{(ijk)_2})$, $\eta_{ijk} \sim \mathcal{L}(e_{(ijk)_1}, e_{(ijk)_2})$. Then model (6.4) can be converted into the following form:

$$\begin{cases} \min & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left(x_{ijk}\frac{d_{(ijk)_1}+d_{(ijk)_2}}{2}+y_{ijk}\frac{e_{(ijk)_1}+e_{(ijk)_2}}{2}\right) \\ s.t. & \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{l}x_{ijk} \le \frac{a_{i_1}+a_{i_2}}{2}, \quad i=1,2,\ldots,m \\ & \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{l}x_{ijk} \ge \frac{b_{j_1}+b_{j_2}}{2}, \quad j=1,2,\ldots,n \\ & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk} \le \frac{c_{k_1}+c_{k_2}}{2}, \quad k=1,2,\ldots,l \\ & x_{ijk} \ge 0, \quad y_{ijk} = \begin{cases} 1, & \text{if } x_{ijk} > 0 \\ 0, & \text{otherwise,} \end{cases} \\ & i=1,2,\ldots,m; \quad j=1,2,\ldots,n; \quad k=1,2,\ldots,l. \end{cases} \tag{6.5}$$

Example 6.2: Suppose the uncertain parameters $\tilde{a}_i$, $\tilde{b}_j$, $\tilde{c}_k$, ξ_{ijk} and η_{ijk} are independent zigzag uncertain variables, i.e., $\tilde{a}_i \sim \mathcal{Z}(a_{i_1}, a_{i_2}, a_{i_3})$, $\tilde{b}_j \sim \mathcal{Z}(b_{j_1}, b_{j_2}, b_{j_3})$, $\tilde{c}_k \sim \mathcal{Z}(c_{k_1}, c_{k_2}, c_{k_3})$, $\xi_{ijk} \sim \mathcal{Z}(d_{(ijk)_1}, d_{(ijk)_2}, d_{(ijk)_3})$, $\eta_{ijk} \sim \mathcal{Z}(e_{(ijk)_1}, e_{(ijk)_2}, e_{(ijk)_3})$. Then model (6.4) can be converted into the following form:

$$\begin{cases} \min & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left(x_{ijk}\frac{d_{(ijk)_1}+2d_{(ijk)_2}+d_{(ijk)_3}}{4}+\right. \\ & \qquad \left. y_{ijk}\frac{e_{(ijk)_1}+2e_{(ijk)_2}+e_{(ijk)_3}}{4}\right) \\ s.t. & \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{l}x_{ijk} \le \frac{a_{i_1}+2a_{i_2}+a_{i_3}}{4}, \quad i=1,2,\ldots,m \\ & \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{l}x_{ijk} \ge \frac{b_{j_1}+2b_{j_2}+b_{j_3}}{4}, \quad j=1,2,\ldots,n \\ & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk} \le \frac{c_{k_1}+2c_{k_2}+c_{k_3}}{4}, \quad k=1,2,\ldots,l \\ & x_{ijk} \ge 0, \quad y_{ijk} = \begin{cases} 1, & \text{if } x_{ijk} > 0 \\ 0, & \text{otherwise,} \end{cases} \\ & i=1,2,\ldots,m; \quad j=1,2,\ldots,n; \quad k=1,2,\ldots,l. \end{cases} \tag{6.6}$$

Example 6.3: Suppose the uncertain parameters $\tilde{a}_i$, $\tilde{b}_j$, $\tilde{c}_k$, ξ_{ijk} and η_{ijk} are independent normal uncertain variables, i.e., $\tilde{a}_i \sim \mathcal{N}(e_{a_i}, \sigma_{a_i})$, $\tilde{b}_j \sim \mathcal{N}(e_{b_j}, \sigma_{b_j})$, $\tilde{c}_k \sim \mathcal{N}(e_{c_k}, \sigma_{c_k})$, $\xi_{ijk} \sim \mathcal{N}(e_{d_{ijk}}, \sigma_{d_{ijk}})$, $\eta_{ijk} \sim \mathcal{N}(e_{t_{ijk}}, \sigma_{t_{ijk}})$. Then model (6.4) can be converted into the following form:

$$\begin{cases} \min & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}\left(x_{ijk}e_{d_{ijk}} + y_{ijk}e_{t_{ijk}}\right) \\ s.t. & \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{l} x_{ijk} \le e_{a_i}, \quad i = 1, 2, \ldots, m \\ & \displaystyle\sum_{i=1}^{m}\sum_{k=1}^{l} x_{ijk} \ge e_{b_j}, \quad j = 1, 2, \ldots, n \\ & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ijk} \le e_{c_k}, \quad k = 1, 2, \ldots, l \\ & x_{ijk} \ge 0, \quad y_{ijk} = \begin{cases} 1, & \text{if } x_{ijk} > 0 \\ 0, & \text{otherwise,} \end{cases} \\ & i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n; \quad k = 1, 2, \ldots, l. \end{cases} \tag{6.7}$$

Lemma 6.1 *(Zhang-Peng-Li-Chen [76]) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent uncertain variables with regular uncertainty distributions $\Phi_1, \Phi_2, \ldots, \Phi_n$, respectively. If $f(x_1, x_2, \ldots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \ldots, x_n$, then $\mathcal{M}\{f(\xi_1, \xi_2, \ldots, \xi_n) \le \bar{f}\} \ge \alpha$ is equivalent to*

$$f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)) \le \bar{f}.$$

Proof: Let $\xi = f(\xi_1, \xi_2, \ldots, \xi_n)$. According to Theorem 2.4, for any $0 < \alpha < 1$, ξ has a regular inverse uncertainty distribution

$$\Psi^{-1}(\alpha) = f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))$$

since $f(x_1, x_2, \ldots, x_n)$ is continuous, strictly increasing with respect to $x_1, x_2, \ldots, x_n$. Then, we have,

$$\mathcal{M}\{\xi \le f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))\} = \mathcal{M}\{\xi \le \Psi^{-1}(\alpha)\} = \alpha.$$

Thus, we get the fact that

$$\mathcal{M}\{\xi \le \bar{f}\} \ge \alpha = \mathcal{M}\{\xi \le f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha))\}.$$

As we know, an uncertainty distribution is an increasing function. This fact tells us that

$$f(\Phi_1^{-1}(\alpha), \Phi_2^{-1}(\alpha), \ldots, \Phi_n^{-1}(\alpha)) \le \bar{f}.$$

The lemma is proved.

Theorem 6.2 *(Zhang-Peng-Li-Chen [76]) If $\tilde{a}_i$, $\tilde{b}_j$, $\tilde{c}_k$, ξ_{ijk} and η_{ijk} are independent uncertain variables with regular uncertainty distributions $\Phi_{\tilde{a}_i}$, $\Phi_{\tilde{b}_j}$, $\Phi_{\tilde{c}_k}$, $\Phi_{\xi_{ijk}}$ and $\Phi_{\eta_{ijk}}$, respectively, then model (6.2) is equivalent to*

$$\begin{cases} \min & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}(x_{ijk}\Phi_{\xi_{ijk}}^{-1}(\alpha)+y_{ijk}\Phi_{\eta_{ijk}}^{-1}(\alpha)) \\ s.t. & \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{l}x_{ijk}-\Phi_{\tilde{a}_i}^{-1}(1-\alpha_i)\le 0, \quad i=1,2,\ldots,m \\ & \displaystyle\Phi_{\tilde{b}_j}^{-1}(\beta_j)-\sum_{i=1}^{m}\sum_{k=1}^{l}x_{ijk}\le 0, \quad j=1,2,\ldots,n \\ & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk}-\Phi_{\tilde{c}_k}^{-1}(1-\gamma_k)\le 0, \quad k=1,2,\ldots,l \\ & x_{ijk}\ge 0, \quad y_{ijk}=\begin{cases} 1, & \textit{if } x_{ijk}>0 \\ 0, & \textit{otherwise,} \end{cases} \\ & i=1,2,\ldots,m; \quad j=1,2,\ldots,n; \quad k=1,2,\ldots,l. \end{cases} \tag{6.8}$$

Proof: According to Lemma 6.1, we can easily prove that model (6.2) is equivalent to the following model

$$\begin{cases} \min & \bar{f} \\ s.t. & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}(x_{ijk}\Phi_{\xi_{ijk}}^{-1}(\alpha)+y_{ijk}\Phi_{\eta_{ijk}}^{-1}(\alpha))\le \bar{f} \\ & \displaystyle\sum_{j=1}^{n}\sum_{k=1}^{l}x_{ijk}-\Phi_{\tilde{a}_i}^{-1}(1-\alpha_i)\le 0, \quad i=1,2,\ldots,m \\ & \displaystyle\Phi_{\tilde{b}_j}^{-1}(\beta_j)-\sum_{i=1}^{m}\sum_{k=1}^{l}x_{ijk}\le 0, \quad j=1,2,\ldots,n \\ & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk}-\Phi_{\tilde{c}_k}^{-1}(1-\gamma_k)\le 0, \quad k=1,2,\ldots,l \\ & x_{ijk}\ge 0, \quad y_{ijk}=\begin{cases} 1, & \text{if } x_{ijk}>0 \\ 0, & \text{otherwise,} \end{cases} \\ & i=1,2,\ldots,m; \quad j=1,2,\ldots,n; \quad k=1,2,\ldots,l. \end{cases} \tag{6.9}$$

Clearly, model (6.9) is equivalent to (6.8). The theorem is proved.

Theorem 6.3 *(Zhang-Peng-Li-Chen [76]) If $\tilde{a}_i$, $\tilde{b}_j$, $\tilde{c}_k$, ξ_{ijk} and η_{ijk} are independent uncertain variables with regular uncertainty distributions $\Phi_{\tilde{a}_i}$, $\Phi_{\tilde{b}_j}$, $\Phi_{\tilde{c}_k}$, $\Phi_{\xi_{ijk}}$ and $\Phi_{\eta_{ijk}}$, respectively, then model (6.3) is equivalent to*

$$\begin{cases} \max & \mathcal{M}\left\{\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{l}(\xi_{ijk}x_{ijk}+\eta_{ijk}y_{ijk}) \le \bar{f}\right\} \\ s.t. & \sum_{j=1}^{n}\sum_{k=1}^{l}x_{ijk} - \Phi_{\tilde{a}_i}^{-1}(1-\alpha_i) \le 0, \quad i=1,2,\ldots,m \\ & \Phi_{\tilde{b}_j}^{-1}(\beta_j) - \sum_{i=1}^{m}\sum_{k=1}^{l}x_{ijk} \le 0, \quad j=1,2,\ldots,n \\ & \sum_{i=1}^{m}\sum_{j=1}^{n}x_{ijk} - \Phi_{\tilde{c}_k}^{-1}(1-\gamma_k) \le 0, \quad k=1,2,\ldots,l \\ & x_{ijk} \ge 0, \quad y_{ijk} = \begin{cases} 1, & \text{if } x_{ijk} > 0 \\ 0, & \text{otherwise,} \end{cases} \\ & i=1,2,\ldots,m; \quad j=1,2,\ldots,n; \quad k=1,2,\ldots,l. \end{cases} \tag{6.10}$$

Proof: The proof is similar to Theorem 6.2, we omit it.

Hybrid Intelligent Algorithm

When the uncertain variables are some special variables such as linear uncertain variables, zigzag uncertain variables, or normal uncertain variables, the expected value model and the chance-constrained programming model can be converted into deterministic forms and can be solved by some traditional methods. Since the uncertain variables in the above models may take any form, it would be difficult to solve the proposed models in traditional ways. For solving the models mentioned above, Zhang-Peng-Li-Chen [76] introduced an hybrid intelligent algorithm which integrate the tabu search algorithm and 99-method. To obtain the uncertainty distribution of a monotone function of uncertain variables, the 99-method was proposed by Liu [47]. With the use of a computer, the 99-method technique is a suitable method to estimate the corresponding values of the uncertain variables. The tabu search algorithm is an artificial intelligence algorithm, which was proposed by Glover [29, 30, 31]. It has successfully applied in many complex optimization problems which are hard to solve by traditional methods.

The 99-Method

To solve the proposed models, we need to compute the expected value $E[\xi_i]$, the inverse uncertainty distribution $\Phi_i^{-1}(\alpha)$ and the uncertain measure

$$\mathcal{M}\left\{\sum_{i=1}^{n}x_i\xi_i \le \bar{f}\right\}.$$

Inspired by the idea of computing these values using the 99-method, Zhang-Peng-Li-Chen [76] designed methods to simulate the corresponding values in the following.

Inverse Uncertainty Distribution: Generally, an uncertain variable ξ_i can be represented in computer as Table 6.1. Then, for any $0 < \alpha < 1$, the inverse uncertainty distribution $\Phi_i^{-1}(\alpha)$ of ξ_i can be estimated by the following formula:

$$\frac{\Phi_i^{-1}(0.01\lfloor 100\alpha \rfloor) + \Phi_i^{-1}(0.01\lceil 100\alpha \rceil)}{2}.$$

Table 6.1: Inverse uncertainty distribution of ξ_i

α	0.01	0.02	0.03	$\cdots$	0.99
$\Phi_i^{-1}(\alpha)$	$\Phi_i^{-1}(0.01)$	$\Phi_i^{-1}(0.02)$	$\Phi_i^{-1}(0.03)$	$\cdots$	$\Phi_i^{-1}(0.99)$

Expected Value: According to Theorem 2.10, the expected value $E[\xi_i]$ can be estimated by

$$\sum_{j=1}^{99} 0.01\Phi_i^{-1}(0.01j).$$

In more detail, $E[\xi_i]$ can be estimated as follows,

Step 1. Set $E = 0$ and $j = 1$.

Step 2. Let $y_j = 0.01\Phi_i^{-1}(0.01j)$, and $E \leftarrow E + y_j$.

Step 3. If $j < 99$, let $j \leftarrow j + 1$. Then turn back to Step 2.

Step 4. Return $E[\xi_i] = E$.

Uncertain Measure: Since

$$f(x_1, x_2, \ldots, x_n, t_1, t_2, \ldots, t_n) = \sum_{i=1}^{n} x_i t_i$$

is a strictly increasing function with respect to $t_1, t_2, \ldots, t_n$. According to Theorem 2.4, the inverse uncertainty distribution of

$$\xi = f(x_1, x_2, \ldots, x_n, \xi_1, \xi_2, \ldots, \xi_n) = \sum_{i=1}^{n} x_i \xi_i$$

can be represented on a computer as Table 6.2. This produces a numerical method for computing uncertain measure

$$\mathcal{M}\left\{\sum_{i=1}^{n} x_i \xi_i \leq \bar{f}\right\}$$

as follows,

Step 1. Set $j = 1$.

Step 2. Let $y_j = f(x_1, x_2, \ldots, x_n, \Phi_1^{-1}(0.01j), \Phi_2^{-1}(0.01j), \ldots, \Phi_n^{-1}(0.01j))$.

Step 3. If $y_j < \bar{f}$, then $j \leftarrow j + 1$, and turn back to Step 2.

Step 4. Report $\alpha = 0.01j$ as the estimation of uncertain measure

$$\mathcal{M}\left\{\sum_{i=1}^{n} x_i\xi_i \leq \bar{f}\right\}.$$

Table 6.2: Inverse uncertainty distribution $\Psi^{-1}(\alpha)$ of ξ

α	$\Psi^{-1}(\alpha)$
0.01	$f(x_1, x_2, \cdots, x_n, \Phi_1^{-1}(0.01), \Phi_2^{-1}(0.01), \ldots, \Phi_n^{-1}(0.01))$
0.02	$f(x_1, x_2, \cdots, x_n, \Phi_1^{-1}(0.02), \Phi_2^{-1}(0.02), \ldots, \Phi_n^{-1}(0.02))$
$\vdots$	$\vdots$
0.99	$f(x_1, x_2, \cdots, x_n, \Phi_1^{-1}(0.99), \Phi_2^{-1}(0.99), \ldots, \Phi_n^{-1}(0.99))$

Solution Representation

Suppose there are m sources, n destinations and l conveyances. Let vector $(t_1, t_2, \ldots, t_{mnl})$ denote a solution $\mathbf{x} = \{x_{ijk}\}$, where $\{t_1, t_2, \ldots, t_{mnl}\}$ is a rearrangement of $\{1, 2, \ldots, mnl\}$. Then a solution $\mathbf{x}$ can be decoded by the following method:

Step 1. Let $s = 1$.

Step 2. If $s \leq mnl$, the index i, j, k for the variable x_{ijk} can be calculated in the following way: Suppose that $t_s = anl + b$, where $0 \leq b < nl$, then

$$i = \begin{cases} a, & \text{if } b = 0 \\ a + 1, & \text{if } 0 < b < nl; \end{cases}$$

Suppose that $b = cl + d$, where $0 \leq d < l$, then

$$j = \begin{cases} n, & \text{if } c = d = 0 \\ c, & \text{if } c \neq 0, d = 0 \\ c + 1, & \text{if } 0 < d < l, \end{cases} \qquad k = \begin{cases} l, & \text{if } d = 0 \\ d, & \text{if } 0 < d < l. \end{cases}$$

Otherwise, stop.

Step 3. Let $e = \min\{a_i, b_j, c_k\}$, where a_i, b_j, and c_k denote supply abilities, demands and conveyance capacities, and are estimated by the 99-method respectively.

Step 4. Let $x_{ijk} = e$, $a_i \leftarrow a_i - e$, $b_j \leftarrow b_j - e$, $c_k \leftarrow c_k - e$; Then let $s \leftarrow s + 1$, go back to Step 2.

After decoding, it is clear that $\mathbf{x}$ is a feasible solution, and the demand of each destination is just satisfied.

Neighborhood Structure

At each iteration, the neighborhood structure of a present optimal solution should be determined by the following way: Any two components in the array exchange their positions. For instance, t_p and t_q are two elements in the solution $\mathbf{x}$. Then a new solution $\mathbf{x}'$ can be obtained by exchange the positions of t_p and t_q. That is,

$$\begin{array}{ccc} \mathbf{x} & \rightarrow & \mathbf{x}' \\ (\ldots, t_p, \ldots, t_q, \ldots) & \rightarrow & (\ldots, t_q, \ldots, t_p, \ldots) \end{array}$$

It is easy to verify that $\mathbf{x}'$ is also a feasible solution. We call that $\mathbf{x}'$ is a feasible neighbor of $\mathbf{x}$. The above exchange process can be described in the following way:

Step 1. Generate two different numbers p and q from the set $\{1, 2, \ldots, mnl\}$ randomly.

Step 2. Exchange the positions of the two elements t_p and t_q.

After repeating the above operation T times, we may obtain the neighborhood $N(\mathbf{x})$ containing T feasible neighbors.

Tabu Moves

In the process of seeking the best solution, a tabu move is made to avoid local recycling. As mentioned above, a new neighbor is produced by exchanging the positions of t_p and t_q in the array. If this neighbor is chosen as the present optimal solution at iteration s, it is forbidden to exchange the positions of t_p and t_q again until iteration $s + g$ (where g is tabu tenure). In order to memory that t_p and t_q are forbidden to exchange their positions, a vector (t_p, t_q) is employed to denote the tabu move.

Aspiration Criterion

Suppose that $\mathbf{x}^*$ is the best solution so far encountered. If a feasible neighbor $\mathbf{x}$ of the present solution made by a tabu move satisfies that the objective value of $\mathbf{x}$ is better than that of $\mathbf{x}^*$, then this tabu move is invalid, that is, solution $\mathbf{x}$ can be selected.

Tabu Search Algorithm

Generally, the procedure of tabu search algorithm can be summarized as follows.

Step 1. Randomly choose a feasible solution $\mathbf{x}$. Let **presentsol** $= \mathbf{x}$, **bestsol** $= \mathbf{x}$, **presentiter** $= 0$, **bestiter** $= 0$, and initialize the tabu list.

Step 2. Produce the neighborhood $N(\mathbf{presentsol})$ with T elements, let **presentiter** $\leftarrow$ **presentiter** $+ 1$.

Step 3. Compute the objective values of elements in $N(\mathbf{presentsol})$ by traditional methods or the 99-method if possible.

Step 4. Find the best solution $\mathbf{x}^*$ in $N(\mathbf{presentsol})$ such that $\mathbf{x}^*$ is non-tabu or satisfies the aspiration criterion.

Step 5. If the objective value of $\mathbf{x}^*$ is better than the objective value of **bestsol**, then let **presentsol** $\leftarrow \mathbf{x}^*$, **bestsol** $\leftarrow \mathbf{x}^*$, **bestiter** $\leftarrow$ **presentiter**; Otherwise, let **presentsol** $\leftarrow \mathbf{x}^*$; Update the tabu list.

Step 6. If **presentiter** $-$ **bestiter** $\leq N$ (where N is the largest number of iteration that objective value of **bestsol** does not change), then go to Step 2; Otherwise, report **bestsol** as an approximate optimal solution, stop.

Numerical Experiments

Suppose that there are four coal mines to supply the coal for six cities, and two kinds of conveyances are available to be selected, i.e., train and cargo ship. In fact, the decision maker cannot get the basic data exactly, since the transportation plan is made in advance. These basic data, such as supply capacity, demand, transportation capacity, are usually obtained by experience evaluation or expert advice. Thus, we assume all expert data are zigzag uncertain variables, which are listed as follows,

$$\tilde{a}_1 \sim \mathcal{Z}(15, 23, 29), \quad \tilde{a}_2 \sim \mathcal{Z}(18, 22, 28),$$

$$\tilde{a}_3 \sim \mathcal{Z}(17, 22, 25), \quad \tilde{a}_4 \sim \mathcal{Z}(20, 23, 26).$$

$$\tilde{b}_1 \sim \mathcal{Z}(5, 8, 10), \quad \tilde{b}_2 \sim \mathcal{Z}(7, 10, 12), \quad \tilde{b}_3 \sim \mathcal{Z}(4, 6, 9),$$

$$\tilde{b}_4 \sim \mathcal{Z}(5, 8, 12), \quad \tilde{b}_5 \sim \mathcal{Z}(10, 13, 15), \quad \tilde{b}_6 \sim \mathcal{Z}(4, 6, 10).$$

$$\tilde{c}_1 \sim \mathcal{Z}(40, 46, 50), \quad \tilde{c}_2 \sim \mathcal{Z}(48, 55, 60).$$

In addition, we assume that the direct costs ξ_{ijk} and fixed charges η_{ijk} are also zigzag uncertain variables, which are listed in Tables 6.3-6.6, respectively.

Table 6.3: The direct costs by train (ξ_{ij1})

Cities (j) \ Mines (i)	1	2	3	4
1	$(5, 7, 8)$	$(3, 6, 8)$	$(4, 5, 8)$	$(8, 10, 12)$
2	$(3, 5, 7)$	$(4, 7, 10)$	$(4, 6, 8)$	$(4, 7, 10)$
3	$(5, 8, 10)$	$(6, 8, 12)$	$(4, 5, 6)$	$(5, 7, 9)$
4	$(4, 6, 10)$	$(4, 5, 9)$	$(8, 9, 10)$	$(4, 6, 8)$
5	$(6, 8, 12)$	$(6, 10, 12)$	$(4, 7, 10)$	$(8, 10, 12)$
6	$(5, 6, 8)$	$(7, 8, 10)$	$(6, 8, 10)$	$(8, 10, 12)$

Table 6.4: The fixed charges by train (η_{ij1})

Cities (j) \ Mines (i)	1	2	3	4
1	$(2, 4, 6)$	$(3, 4, 5)$	$(2, 3, 4)$	$(2, 4, 6)$
2	$(4, 5, 6)$	$(1, 2, 3)$	$(3, 4, 5)$	$(1, 2, 4)$
3	$(2, 4, 6)$	$(3, 4, 5)$	$(2, 3, 4)$	$(1, 2, 3)$
4	$(1, 3, 5)$	$(2, 3, 4)$	$(3, 5, 7)$	$(2, 3, 4)$
5	$(3, 4, 5)$	$(4, 5, 6)$	$(2, 4, 6)$	$(2, 4, 6)$
6	$(2, 3, 4)$	$(1, 3, 5)$	$(2, 3, 4)$	$(4, 5, 6)$

Table 6.5: The direct costs by cargo ship (ξ_{ij2})

Cities (j) \ Mines (i)	1	2	3	4
1	$(11, 13, 15)$	$(6, 8, 10)$	$(8, 10, 12)$	$(6, 8, 10)$
2	$(8, 10, 12)$	$(7, 10, 13)$	$(8, 11, 14)$	$(6, 9, 12)$
3	$(6, 8, 10)$	$(5, 9, 13)$	$(6, 10, 14)$	$(11, 13, 15)$
4	$(7, 10, 13)$	$(10, 12, 14)$	$(6, 7, 8)$	$(8, 10, 12)$
5	$(10, 12, 14)$	$(10, 13, 16)$	$(11, 12, 13)$	$(5, 10, 15)$
6	$(8, 10, 12)$	$(9, 12, 15)$	$(8, 9, 10)$	$(6, 8, 10)$

Table 6.6: The fixed charges by cargo ship (η_{ij2})

Cities (j) \ Mines (i)	1	2	3	4
1	$(5, 6, 7)$	$(3, 4, 5)$	$(4, 5, 6)$	$(1, 3, 5)$
2	$(2, 4, 6)$	$(5, 6, 7)$	$(1, 4, 7)$	$(3, 5, 7)$
3	$(4, 5, 6)$	$(2, 4, 6)$	$(4, 5, 6)$	$(5, 6, 7)$
4	$(3, 5, 7)$	$(5, 6, 7)$	$(1, 2, 3)$	$(3, 5, 7)$
5	$(4, 6, 8)$	$(2, 5, 8)$	$(4, 6, 8)$	$(1, 4, 7)$
6	$(3, 5, 7)$	$(1, 4, 7)$	$(1, 4, 7)$	$(1, 3, 5)$

Example 6.4: If we want to obtain the expected transportation plan of the uncertain fixed charge solid transportation problem, then the expected value model can be formulated as follows,

$$\begin{cases} \min & E\left[\sum_{i=1}^{4}\sum_{j=1}^{6}\sum_{k=1}^{2}(x_{ijk}\xi_{ijk}+y_{ijk}\eta_{ijk})\right] \\ s.t. & E\left[\sum_{j=1}^{6}\sum_{k=1}^{2}x_{ijk}-\tilde{a}_i\right]\le 0, \quad i=1,2,\ldots,4 \\ & E\left[\sum_{i=1}^{4}\sum_{k=1}^{2}x_{ijk}-\tilde{b}_j\right]\ge 0, \quad j=1,2,\ldots,6 \\ & E\left[\sum_{i=1}^{4}\sum_{j=1}^{6}x_{ijk}-\tilde{c}_k\right]\le 0, \quad k=1,2 \\ & x_{ijk}\ge 0, \quad y_{ijk}=\begin{cases} 1, & \text{if } x_{ijk}>0 \\ 0, & \text{otherwise,} \end{cases} \\ & i=1,2,\ldots,4; \quad j=1,2,\ldots,6; \quad k=1,2. \end{cases} \tag{6.11}$$

We set the parameters in the tabu search algorithm as follows: the number of elements in the neighborhood $T = 30$, the largest number of iterations that the objective value of the best solution does not change $N = 500$, tabu tenure $g = 10$. Using a person computer generates the following optimal plan is $x_{121} = 9.75$, $x_{211} = 7.75$, $x_{241} = 8.25$, $x_{331} = 6.25$, $x_{351} = 12.75$, $x_{462} = 6.5$, whose objective value is 335.25.

Generally, different parameters in the algorithm may result in different optimal objective values. Thus, we make more numerical experiments with different parameter values in the tabu search algorithm and propose a relative error to compare the results of the objective values. The relative error is computed by the following formula:

$$\text{relative error} = \frac{\text{objective value - the best objective value}}{\text{the best objective value}} \times 100\%.$$

The values of the parameters and the experiment results are shown in Table 6.7. It can be seen from Table 6.7 that the relative errors do not exceed 2.55%. This fact illustrates that the proposed hybrid intelligent algorithm is stable and robust.

Example 6.5: Let the decision maker set the goal as the minimization of the total cost at a confidence level α not less than 0.9. We also set $\alpha_i = \beta_j = \gamma_k = 0.9$, where $i = 1, 2, \ldots, 4$; $j = 1, 2, \ldots, 6$; $k = 1, 2$. Then a chance-constrained programming model for the uncertain fixed charge solid

Table 6.7: Comparison of the objective values of model (6.11)

g	T	N	objective value	relative error
5	10	50	341.69	1.92%
5	20	100	343.81	2.55%
5	30	300	335.25	0.00%
10	10	200	335.25	0.00%
10	30	500	335.25	0.00%
10	50	600	335.25	0.00%

transportation problem is established as follows,

$$\begin{cases} \min \quad \bar{f} \\ s.t. \quad \mathcal{M}\left\{\sum\limits_{i=1}^{4}\sum\limits_{j=1}^{6}\sum\limits_{k=1}^{2}(\xi_{ijk}x_{ijk}+\eta_{ijk}y_{ijk}) \le \bar{f}\right\} \ge 0.9 \\ \quad\quad \mathcal{M}\left\{\sum\limits_{j=1}^{6}\sum\limits_{k=1}^{2}x_{ijk} \le \tilde{a}_i\right\} \ge 0.9, \quad i=1,2,\ldots,4 \\ \quad\quad \mathcal{M}\left\{\sum\limits_{i=1}^{4}\sum\limits_{k=1}^{2}x_{ijk} \ge \tilde{b}_j\right\} \ge 0.9, \quad j=1,2,\ldots,6 \\ \quad\quad \mathcal{M}\left\{\sum\limits_{i=1}^{4}\sum\limits_{j=1}^{6}x_{ijk} \le \tilde{c}_k\right\} \ge 0.9, \quad k=1,2 \\ \quad\quad x_{ijk} \ge 0, \quad y_{ijk} = \begin{cases} 1, & \text{if } x_{ijk} > 0 \\ 0, & \text{otherwise,} \end{cases} \\ \quad\quad i=1,2,\ldots,4; \quad j=1,2,\ldots,6; \quad k=1,2. \end{cases} \tag{6.12}$$

Set $T = 30$, $N = 500$, and tabu tenure $g = 10$. After running on a personal computer, the proposed hybrid intelligent algorithm tells us that the optimal plan of model (6.12) is $x_{121} = 11.60$, $x_{331} = 3.40$, $x_{351} = 14.60$, $x_{431} = 0.20$, $x_{441} = 11.20$, $x_{132} = 4.80$, $x_{212} = 9.60$, $x_{462} = 9.20$, whose objective value is 583.92.

Example 6.6: In fact, the decision maker may firstly present a satisfying maximal cost $\bar{f}$, and then prefers finding a transportation plan that maximizes the uncertain measure that the total cost is no more than the given value. Suppose that the satisfying predetermined maximal cost is $\bar{f} = 600$. We also assume that $\alpha_i = \beta_j = \gamma_k = 0.9$, where $i = 1, 2, \ldots, 4$; $j = 1, 2, \ldots, 6$; $k = 1, 2$. Then the measure-chance programming may be presented as fol-

lows:

$$\begin{cases} \max & \mathcal{M}\left\{\sum_{i=1}^{4}\sum_{j=1}^{6}\sum_{k=1}^{2}(\xi_{ijk}x_{ijk}+\eta_{ijk}y_{ijk}) \le 600\right\} \\ s.t. & \mathcal{M}\left\{\sum_{j=1}^{6}\sum_{k=1}^{2}x_{ijk} \le \tilde{a}_i\right\} \ge 0.9, \quad i=1,2,\ldots,4 \\ & \mathcal{M}\left\{\sum_{i=1}^{4}\sum_{k=1}^{2}x_{ijk} \ge \tilde{b}_j\right\} \ge 0.9, \quad j=1,2,\ldots,6 \\ & \mathcal{M}\left\{\sum_{i=1}^{4}\sum_{j=1}^{6}x_{ijk} \le \tilde{c}_k\right\} \ge 0.9, \quad k=1,2 \\ & x_{ijk} \ge 0, \quad y_{ijk} = \begin{cases} 1, & \text{if } x_{ijk} > 0 \\ 0, & \text{otherwise,} \end{cases} \\ & i=1,2,\ldots,4; \quad j=1,2,\ldots,6; \quad k=1,2. \end{cases} \tag{6.13}$$

Model (6.13) is complex, and is difficult to solve it by traditional methods. Fortunately, we can solve it by using the proposed hybrid intelligent algorithm. We also set $T = 30$, $N = 500$, and tabu tenure $g = 10$. Using a person computer generates the following optimal plan is $x_{121} = 11.60$, $x_{331} = 3.40$, $x_{351} = 14.60$, $x_{441} = 11.20$, $x_{132} = 5.00$, $x_{212} = 9.60$, $x_{462} = 9.20$, whose objective value is 0.96.

6.2 Dispatching Medical Supplies

Dispatching medical supplies is one of the fundamental problems in emergency events. The problem can be simply described as follows: Suppose that in a central depot (e.g. airport or central inventory), m vehicles are available for n demanding locations. The problem involves to plan the routes such that medical supplies are transported by vehicles to the demanding locations as soon as possible once an emergency event occurs. The problem of dispatching medical supplies in an uncertain environment was first studied by Li-Peng-Li-Su [40].

For simplicity, we assume that (a) each vehicle has a container with a physical limitation so that the total loading of each vehicle cannot exceed its capacity; (b) a vehicle will be assigned for only one route at most one time; (c) each route begins and ends at the central depot; (d) each demanding location will be serviced by one and only one vehicle; and (e) ignore the unload time by setting it a zero.

Because of the highly unpredictable nature of emergencies and the severity of the accident, the running times and the demands usually cannot be known precisely. Generally, we have no choice but to invite some domain experts

to evaluate the belief degrees about them. In this case, it is reasonable to describe the running times and demands as uncertain variables.

Let us first introduce the following notations and assumptions:

Notations

$i = 0$: Depot;

$i = 1, 2, \ldots, n$: Demanding locations;

$k = 1, 2, \ldots, m$: Vehicles;

ξ_{ij} : Uncertain running time from demanding locations i to j, $i, j = 0, 1, 2, \ldots, n$;

η_i : Uncertain demand of demanding location i, $i = 1, 2, \ldots, n$;

C_k : Capacity of vehicle k, $k = 1, 2, \ldots, m$.

Operational Plan

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$: Integer decision vector representing n demanding locations with $1 \leq x_i \leq n$ and $x_i \neq x_j$ for all $i \neq j$, $i, j = 1, 2, \ldots, n$;

$\mathbf{y} = (y_1, y_2, \ldots, y_{m-1})$: Integer decision vector with $y_0 = 0 \leq y_1 \leq y_2 \leq \cdots \leq y_{m-1} \leq n = y_m$.

We note that the operational plan is fully determined by the decision vectors $\mathbf{x}$ and $\mathbf{y}$ in the following way:

For each $k (1 \leq k \leq m)$, if $y_k = y_{k-1}$, then vehicle k is not used; Otherwise, vehicle k is used, and the tour of vehicle k is:

$$0 \to x_{y_{k-1}+1} \to x_{y_{k-1}+2} \to \cdots \to x_{y_k} \to 0.$$

Vehicle Routing Models

Generally speaking, emergencies may lead to the scarcity of relief supplies. The decision maker would accept to some extant that the quantity of the supplies delivered by a vehicle cannot reach the total demand in the corresponding route. However, at a given confidence level considered as the safety margin, the demand must be achieved. Suppose that the total amount of the medical supplies transported by each vehicle equals its capacity. The satisfaction constraint can be formulated as follows: For $1 \leq k \leq m$, if $y_k > y_{k-1}$, then

$$\mathcal{M}\left\{\sum_{j=y_{k-1}+1}^{y_k} \eta_{x_j} \leq C_k\right\} \geq \alpha$$

where α is the predetermined confidence level.

Additionally, we set $\boldsymbol{\xi} = \{\xi_{ij} : i, j = 0, 1, 2, \ldots, n\}$. Denote $f_k(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi})$ as the total running time function of vehicle k in the corresponding route for $k = 1, 2, \ldots, m$.

According to the assumptions, the total running time of vehicle k can be obtained. That is,

$$f_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) = \begin{cases} \xi_{0x_{y_{k-1}+1}} + \sum\limits_{i=y_{k-1}+1}^{y_k-1} \xi_{x_i x_{i+1}} + \xi_{x_{y_k}0}, & \text{if } y_k > y_{k-1} \\ 0, & \text{if } y_k = y_{k-1}. \end{cases}$$

Obviously, $f_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$ is also an uncertain variable. In order to rank different uncertain variables, some effective criteria should be presented. The expected value criterion is a common way to be employed to rank uncertain variables. That is, the larger the expected value is, the larger the corresponding uncertain variable is. Based on expected value criterion, the goal of the problem can be formulated as

$$\min_{\mathbf{x},\mathbf{y}} \max_{1\le k\le m} E[f_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi})].$$

Consequently, if the decision maker prefers to solve the problem in the sense of expected value criterion, a mathematical model for the problem of dispatching medical supplies in emergency can be constructed as follows (Li-Peng-Li-Su [40]),

$$\begin{cases} \min\limits_{\mathbf{x},\mathbf{y}} \max\limits_{1\le k\le m} E[f_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi})] \\ \text{subject to :} \\ \quad \mathcal{M}\left\{\sum\limits_{j=y_{k-1}+1}^{y_k} \eta_{x_j} \le C_k\right\} \ge \alpha, \ \text{ if } y_k > y_{k-1} \\ \quad k = 1,2,\ldots,m \\ \quad 1 \le x_i \le n, \quad i = 1,2,\ldots,n \\ \quad x_i \ne x_j, \quad i \ne j, \quad i,j = 1,2,\ldots,n \\ \quad 0 \le y_1 \le y_2 \le \cdots \le y_{m-1} \le n \\ \quad x_i, \quad i = 1,2,\ldots,n, \quad \text{integers} \\ \quad y_j, \quad j = 1,2,\ldots,m-1, \quad \text{integers} \end{cases} \tag{6.14}$$

where α is a predetermined confidence level.

On the other hand, the decision maker may firstly present a satisfying predetermined maximal running time $\bar{f}$, and then maximize the minimum uncertain measure that the running time $f_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi})$ is no more than the given value. If this modeling idea is approved, a measure-chance programming

model for the problem can be constructed as follows (Li-Peng-Li-Su [40]),

$$\begin{cases} \max\limits_{\mathbf{x},\mathbf{y}} \min\limits_{1\le k\le m} \mathcal{M}\{f_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) \le \bar{f}\} \\ \text{subject to :} \\ \quad \mathcal{M}\left\{\sum\limits_{j=y_{k-1}+1}^{y_k} \eta_{x_j} \le C_k\right\} \ge \alpha, \ \text{ if } y_k > y_{k-1} \\ \quad k = 1,2,\ldots,m \\ \quad 1 \le x_i \le n, \quad i = 1,2,\ldots,n \\ \quad x_i \ne x_j, \quad i \ne j, \quad i,j = 1,2,\ldots,n \\ \quad 0 \le y_1 \le y_2 \le \cdots \le y_{m-1} \le n \\ \quad x_i, \quad i = 1,2,\ldots,n, \quad \text{integers} \\ \quad y_j, \quad j = 1,2,\ldots,m-1, \quad \text{integers} \end{cases} \tag{6.15}$$

where α is a predetermined confidence level.

Discussion of the Models

Note that there are many uncertain variables in the proposed models. For seeking the optimal solution for the models, it is necessary for us to obtain the values of the expected value and uncertain measure. Generally, for the convenience of calculation, it is natural for us to discuss the equivalents of the models.

Theorem 6.1 *(Li-Peng-Li-Su [40]) If ξ_{ij} are independent uncertain variables with uncertainty distributions Φ_{ij}, $i,j = 0,1,2,\ldots,n$, then*

$$E[f_k(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi})] = \begin{cases} \displaystyle\int_0^1 \Phi^{-1}_{0x_{y_{k-1}+1}}(\alpha)\mathrm{d}\alpha + \sum_{i=y_{k-1}+1}^{y_k-1} \int_0^1 \Phi^{-1}_{x_i x_{i+1}}(\alpha)\mathrm{d}\alpha + \\ \displaystyle\int_0^1 \Phi^{-1}_{x_{y_k}0}(\alpha)\mathrm{d}\alpha, & \text{if } y_k > y_{k-1} \\ 0, & \text{if } y_k = y_{k-1}, \end{cases}$$

for $k = 1,2,\ldots,m$.

Proof: Since ξ_{ij}, $i,j = 0,1,2,\ldots,n$, are independent uncertain variables, it follows from the linearity of expected value operator of uncertain variable that, for any k $(1 \le k \le m)$,

$$E[f_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi})] = \begin{cases} E\left[\xi_{0x_{y_{k-1}+1}}\right] + \displaystyle\sum_{i=y_{k-1}+1}^{y_k-1} E\left[\xi_{x_i x_{i+1}}\right] + \\ E\left[\xi_{x_{y_k}0}\right], & \text{if } \ y_k > y_{k-1} \\ 0, & \text{if } \ y_k = y_{k-1}. \end{cases}$$

According to Theorem 2.10, the theorem can be easily proved.

For any $(\mathbf{x}, \mathbf{y})$, if $y_k > y_{k-1}$, $k = 1, 2, \ldots, m$, we need to check whether it satisfies the following chance constraint:

$$\mathcal{M}\left\{\sum_{j=y_{k-1}+1}^{y_k} \eta_{x_j} \leq C_k\right\} \geq \alpha. \tag{6.16}$$

Li-Peng-Li-Su [40] proved that it can be transformed into the corresponding crisp equivalent if the uncertain variables η_i are independent uncertain variables, $i = 1, 2, \ldots, n$.

Theorem 6.2 *(Li-Peng-Li-Su [40]) If η_i are independent uncertain variables with regular uncertainty distributions Ψ_i, $i = 1, 2, \ldots, n$, then chance constraint (6.16) can be transformed into*

$$\sum_{j=y_{k-1}+1}^{y_k} \Psi_{x_j}^{-1}(\alpha) \leq C_k,$$

$k = 1, 2, \ldots, m$.

Proof: Let

$$\tau_k = g_k(\eta_{x_{y_{k-1}+1}}, \eta_{x_{y_{k-1}+2}}, \ldots, \eta_{x_{y_k}}) = \sum_{j=y_{k-1}+1}^{y_k} \eta_{x_j}$$

for $k = 1, 2, \ldots, m$. Denote Θ_k as the uncertainty distributions of τ_k, $k = 1, 2, \ldots, m$. Since η_i are independent uncertain variables, it follows from Theorem 2.4 that, for any $0 < \alpha < 1$, we have

$$\Theta_k^{-1}(\alpha) = \sum_{j=y_{k-1}+1}^{y_k} \Psi_{x_j}^{-1}(\alpha), \quad k = 1, 2, \ldots, m.$$

This means that

$$\mathcal{M}\left\{\tau_k \leq \sum_{j=y_{k-1}+1}^{y_k} \Psi_{x_j}^{-1}(\alpha)\right\} = \mathcal{M}\left\{\tau_k \leq \Theta_k^{-1}(\alpha)\right\} = \alpha.$$

Since the uncertainty distribution is an increasing function, for any k $(1 \leq k \leq m)$, from

$$\mathcal{M}\left\{\tau_k \leq C_k\right\} \geq \alpha = \mathcal{M}\left\{\tau_k \leq \sum_{j=y_{k-1}+1}^{y_k} \Psi_{x_j}^{-1}(\alpha)\right\},$$

we know

$$C_k \geq \sum_{j=y_{k-1}+1}^{y_k} \Psi_{x_j}^{-1}(\alpha).$$

The proof is completed.

Clearly, for different confidence levels α, we may obtain different optimal objective values for model (6.14). How about the relation between the optimal objective value and α? The following theorem answers this question.

Theorem 6.3 *(Li-Peng-Li-Su [40]) If η_i are independent uncertain variables with regular uncertainty distributions Ψ_i, $i = 1, 2, \ldots, n$, then the optimal objective value of model (6.14) is non-decreasing with respect to confidence level α.*

Proof: Denote $F(\alpha)$ as the feasible set of the constraint (6.16) with respect to α, and $Opti(\alpha)$ as the corresponding optimal objective value.

Without loss of generality, we assume that $\alpha_1 \geq \alpha_2$. Then, we have

$$\Theta_k^{-1}(\alpha_1) \geq \Theta_k^{-1}(\alpha_2),$$

that is,

$$\sum_{j=y_{k-1}+1}^{y_k} \Psi_{x_j}^{-1}(\alpha_1) \geq \sum_{j=y_{k-1}+1}^{y_k} \Psi_{x_j}^{-1}(\alpha_2).$$

According to Theorem 6.2, we have $F(\alpha_1) \subseteq F(\alpha_2)$. Additionally, the optimal objective value of model (6.14) with respect to α_1 is greater than or equal to that with respect to α_2, i.e., $Opti(\alpha_1) \geq Opti(\alpha_2)$. The theorem is proved.

As the similar proof of Theorem 6.3, Li-Peng-Li-Su [40] also proved the following corollary.

Corollary 6.1 *(Li-Peng-Li-Su [40]) If η_i are independent uncertain variables with regular uncertainty distributions Ψ_i, $i = 1, 2, \ldots, n$, then the optimal objective value of model (6.15) is non-increasing with respect to confidence level α.*

Hybrid Intelligent Algorithm

Generally, the proposed models cannot be solved by traditional methods. We need to find some effective algorithms to solve them in general cases. As we know, genetic algorithms have successfully solved many complex optimization problems that are difficult to solve using traditional methods. Li-Peng-Li-Su [40] designed a hybrid intelligent algorithm based on uncertain simulations and genetic algorithms to solve the proposed models. The introduction of the algorithm is as follows.

Uncertain Simulation

Liu [47] pointed out that an uncertain variable ξ_i with uncertainty distribution Φ_i can be represented as Table 6.8 according to expert estimation.

Table 6.8: Uncertainty distribution of uncertain variable ξ_i

α	0.01	0.02	$\cdots$	0.99
$\Phi_i^{-1}(\alpha)$	t_i^1	t_i^2	$\cdots$	t_i^{99}

In Table 6.8, $t_i^1, t_i^2, \ldots, t_i^{99}$ in the second row are the values of inverse uncertainty distribution for $\Phi_i^{-1}(0.01), \Phi_i^{-1}(0.02), \ldots, \Phi_i^{-1}(0.99)$.

Expected Value

It follows from Table 6.8 that Li-Peng-Li-Su [40] estimated the expected value

$$E[\xi_i] = \int_0^1 \Phi_i^{-1}(\alpha)\mathrm{d}\alpha$$

by trapezoid as follows,

$$\begin{aligned} \int_0^1 \Phi_i^{-1}(\alpha)\mathrm{d}\alpha &\approx \sum_{s=1}^{98} \frac{0.01[\Phi_i^{-1}(0.01s) + \Phi_i^{-1}(0.01(s+1))]}{2} \\ &= \sum_{s=1}^{98} \frac{t_i^s + t_i^{s+1}}{200}. \end{aligned} \tag{6.17}$$

Then, Li-Peng-Li-Su [40] introduced an approximate method to compute $E[\xi_i]$ as follows,

Step 1. Set $E = 0$, and $s = 1$.

Step 2. Let $y_s = t_i^s + t_i^{s+1}$, and $E \leftarrow E + y_s$.

Step 3. If $s < 98$, let $s \leftarrow s + 1$, and then turn back to Step 2.

Step 4. Report $E/200$ as the estimation of $E[\xi_i]$.

According to the algorithm designed above, the expected value $E[f_k(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi})]$ also can be estimated.

Inverse Uncertainty Distribution

According to Table 6.8, Li-Peng-Li-Su [40] proposed a method to obtain the inverse uncertainty distribution at any α $(0 < \alpha < 1)$ as follows,

$$\Phi_i^{-1}(\alpha) \approx \frac{\Phi_i^{-1}(0.01\lfloor 100\alpha \rfloor) + \Phi_i^{-1}(0.01\lceil 100\alpha \rceil)}{2} = \frac{t_i^{\lfloor 100\alpha \rfloor} + t_i^{\lceil 100\alpha \rceil}}{2}.$$

Therefore, we can obtain the inverse uncertainty distribution $\Psi_{x_j}^{-1}(\alpha)$ by the same way.

Uncertain Measure

For model (6.15), we need to obtain the value of the uncertain measure

$$\mathcal{M}\{f_k(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) \leq \bar{f}\}$$

for any feasible solution $(\mathbf{x}, \mathbf{y})$. For each k $(1 \leq k \leq m)$, if $y_k > y_{k-1}$, it is clear that $f_k(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi})$ is a continuous, strictly increasing function respect to $\xi_{0x_{y_{k-1}+1}}, \xi_{x_{y_{k-1}+1}x_{y_{k-1}+2}}, \ldots, \xi_{x_{y_k}0}$. Suppose that ξ_i are independent uncertain variables with uncertainty distributions Φ_i, which can be presented similarly as Table 6.8, for $i = 1, 2, \ldots, n$. Let $g(x_1, x_2, \ldots, x_n)$ be a continuous, strictly increasing function with respect to $x_1, x_2, \ldots, x_n$. It follows from Theorem 2.4 that the inverse uncertainty distribution $\Psi^{-1}(\alpha)$ of $g(\xi_1, \xi_2, \ldots, \xi_n)$ can be expressed as Table 6.9.

Table 6.9: Inverse uncertainty distribution $\Psi^{-1}(\alpha)$ of $g(\xi_1, \xi_2, \ldots, \xi_n)$

α	0.01	0.02	...	0.99
$\Psi^{-1}(\alpha)$	$g(t_1^1, t_2^1, \ldots, t_n^1)$	$g(t_1^2, t_2^2, \ldots, t_n^2)$	...	$g(t_1^{99}, t_2^{99}, \ldots, t_n^{99})$

Then, Li-Peng-Li-Su [40] designed the process to simulate the uncertain measure of $\mathcal{M}\{g(\xi_1, \xi_2, \ldots, \xi_n) \leq \bar{f}\}$:

Step 1. Set $i = 1$.

Step 2. Let $y_i = g(t_1^i, t_2^i, \ldots, t_n^i)$.

Step 3. If $y_i < \bar{f}$, let $i \leftarrow i + 1$, and then turn back to Step 2.

Step 4. Report $\alpha = 0.01i$ as the estimation of $\mathcal{M}\{g(\xi_1, \xi_2, \ldots, \xi_n) \leq \bar{f}\}$.

Similarly, we can estimate the value of

$$\mathcal{M}\{f_k(\mathbf{x}, \mathbf{y}, \boldsymbol{\xi}) \leq \bar{f}\}$$

by the algorithm mentioned above.

Representation

Suppose that there are n demanding locations and m vehicles. In the genetic algorithm, Li-Peng-Li-Su [40] introduced chromosomes $S = (s_1, s_2, \ldots, s_n)$ and $T = (t_1, t_2, \ldots, t_{m-1})$ to represent decision vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_{m-1})$, respectively. The matching between the solution vector $(\mathbf{x}, \mathbf{y})$ and the chromosome vector (S, T) is $\mathbf{x} \equiv S$, $\mathbf{y} \equiv T$.

Initialization Process

When we initialize pop_size feasible chromosomes, the feasibility of each chromosome should be checked.

Firstly, an integer vector $S = (s_1, s_2, \ldots, s_n)$ is generated from integer set $\{1, 2, \ldots, n\}$ such that $1 \leq s_i \leq n$ and $s_i \neq s_j$ for $i \neq j$, $i, j = 1, 2, \ldots, n$.

Also, an integer vector $T = (t_1, t_2, \ldots, t_{m-1})$, where $0 \leq t_1 \leq t_2 \leq \cdots \leq t_{m-1} \leq n$, is generated from integer set $\{0, 1, 2, \ldots, n\}$. Then the feasibility of (S, T) is checked by uncertain simulation as follows,

If $y_1 > 0, \sum_{j=1}^{y_1} \Psi_{x_j}^{-1}(\alpha) > C_1$, return 0;

If $y_2 > y_1, \sum_{j=y_1+1}^{y_2} \Psi_{x_j}^{-1}(\alpha) > C_2$, return 0;

...

If $y_m > y_{m-1}, \sum_{j=y_{m-1}+1}^{y_m} \Psi_{x_j}^{-1}(\alpha) > C_m$, return 0;

Otherwise, return 1;

in which 1 means feasible, and 0 non-feasible. If (S, T) is feasible, we take it as an initial chromosome vector accordingly. Otherwise, another integer vector (S, T) is generated until (S, T) is proved to be feasible and taken as an initial chromosome vector. Suppose that

$$(S_1, T_1), (S_2, T_2), \ldots, (S_{pop\_size}, T_{pop\_size})$$

are produced for initial feasible chromosomes by repeating the above process pop_size times.

Genetic Algorithm

After crossover, mutation, and selection, the new population is ready for its next evaluation. Now, we can summarize the procedure of the genetic algorithm for solving the proposed uncertain models as follows.

Step 1. Initialize pop_size chromosome vectors at random.

Step 2. Update the chromosome vectors by crossover and mutation operations.

Step 3. Calculate the objective values of the chromosome vectors by uncertain simulation or the traditional method.

Step 4. Rearrange the chromosome vectors according to the objective values.

Step 5. Compute the fitness of each chromosome vector according to the rank-based evaluation function.

Step 6. Select the chromosome vectors by spanning the roulette wheel.

Step 7. Repeat Step 2 to Step 6 for a given number of cycles.

Step 8. Take the best chromosome vector as an approximate optimal solution.

Table 6.10: The running time ξ_{ij}

Location (i) \ Location (j)	0	1	2	3
0	0	$\mathcal{L}(2,4)$	$\mathcal{L}(4,7)$	$\mathcal{L}(3,6)$
1	$\mathcal{L}(3,5)$	0	$\mathcal{L}(5,8)$	$\mathcal{L}(5,7)$
2	$\mathcal{L}(2,6)$	$\mathcal{L}(2,5)$	0	$\mathcal{L}(4,7)$
3	$\mathcal{L}(3,6)$	$\mathcal{L}(3,5)$	$\mathcal{L}(6,8)$	0
4	$\mathcal{L}(5,7)$	$\mathcal{L}(3,6)$	$\mathcal{L}(6,9)$	$\mathcal{L}(3,6)$
5	$\mathcal{L}(2,5)$	$\mathcal{L}(7,9)$	$\mathcal{L}(5,8)$	$\mathcal{L}(2,5)$
6	$\mathcal{L}(5,9)$	$\mathcal{L}(4,6)$	$\mathcal{L}(5,7)$	$\mathcal{L}(3,5)$
7	$\mathcal{L}(2,5)$	$\mathcal{L}(5,8)$	$\mathcal{L}(3,6)$	$\mathcal{L}(3,5)$

Location (i) \ Location (j)	4	5	6	7
0	$\mathcal{L}(4,7)$	$\mathcal{L}(3,5)$	$\mathcal{L}(5,8)$	$\mathcal{L}(4,6)$
1	$\mathcal{L}(5,7)$	$\mathcal{L}(4,6)$	$\mathcal{L}(4,7)$	$\mathcal{L}(3,5)$
2	$\mathcal{L}(3,6)$	$\mathcal{L}(6,8)$	$\mathcal{L}(5,7)$	$\mathcal{L}(4,7)$
3	$\mathcal{L}(4,6)$	$\mathcal{L}(3,7)$	$\mathcal{L}(3,6)$	$\mathcal{L}(2,5)$
4	0	$\mathcal{L}(3,5)$	$\mathcal{L}(4,7)$	$\mathcal{L}(5,8)$
5	$\mathcal{L}(4,6)$	0	$\mathcal{L}(3,6)$	$\mathcal{L}(5,7)$
6	$\mathcal{L}(4,7)$	$\mathcal{L}(4,6)$	0	$\mathcal{L}(4,6)$
7	$\mathcal{L}(2,5)$	$\mathcal{L}(3,6)$	$\mathcal{L}(3,6)$	0

Numerical Examples

Suppose that there are 3 vehicles are available in a central depot (airport) for 7 demanding locations. Let us show the applications of the proposed models and test the effectiveness of the designed hybrid intelligent algorithm.

As mentioned before, the highly unpredictable nature of emergencies may lead to uncertainty both in demands and running times. In this case, the usual way is to obtain the uncertain data by means of experience evaluation or expert advice. In the following examples, the running times $\xi_{ij}(i,j = 0,1,2,\ldots,7)$ and the demands $\eta_i(i = 1,2,\ldots,7)$ are assumed to be independent linear uncertain variables listed in Tables 6.10 and 6.11, respectively. We also assume that the capacities of the vehicles are $C_1 = 38$, $C_2 = 46$, $C_3 = 50$, respectively.

Example 6.7: Suppose that the satisfaction constraint for each route should hold at confidence level $\alpha = 0.8$. If the decision maker prefers optimizing the objective function in the sense of expected running time, then an uncertain

Table 6.11: The demand η_i of demanding location i

Location i	Demand η_i
1	$\mathcal{L}(12, 18)$
2	$\mathcal{L}(10, 14)$
3	$\mathcal{L}(13, 18)$
4	$\mathcal{L}(10, 14)$
5	$\mathcal{L}(15, 19)$
6	$\mathcal{L}(12, 16)$
7	$\mathcal{L}(13, 16)$

programming model can be constructed as follows,

$$\begin{cases} \min\limits_{\mathbf{x},\mathbf{y}} \quad \max\limits_{1\le k\le 3} E\left[f_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi})\right] \\ \text{subject to :} \\ \quad \mathcal{M}\left\{\sum\limits_{j=y_{k-1}+1}^{y_k} \eta_{x_j} \le C_k\right\} \ge 0.8, \ \text{ if } \ y_k > y_{k-1} \\ \quad k = 1, 2, 3 \\ \quad 1 \le x_i \le 7, \quad i = 1, 2, \ldots, 7 \\ \quad x_i \ne x_j, \quad i \ne j, \quad i, j = 1, 2, \ldots, 7 \\ \quad 0 \le y_1 \le y_2 \le 7 \\ \quad x_i, \quad i = 1, 2, \ldots, 7, \quad \text{integers} \\ \quad y_j, \quad j = 1, 2, \quad \text{integers.} \end{cases} \tag{6.18}$$

Set the crossover probability $P_c = 0.5$, the mutation probability $P_m = 0.3$, the population size $pop\_size = 10$, and the parameter in the rank-based evaluation function is 0.06. After running for 1000 generations, the proposed algorithm shows that the optimal solution is

$$\begin{aligned} &\mathbf{x}^*\text{=(3,4,6,5,1,7,2),} \\ &\mathbf{y}^*\text{=(2,4).} \end{aligned}$$

In other words, the optimal operational plan is

$$\begin{aligned} &\text{Vehicle 1: Airport} \to 3 \to 4 \to \text{Airport} \\ &\text{Vehicle 2: Airport} \to 6 \to 5 \to \text{Airport} \\ &\text{Vehicle 3: Airport} \to 1 \to 7 \to 2 \to \text{Airport} \end{aligned}$$

whose objective value is 15.5.

Generally, if we change the parameters in the algorithm, the objective value will be different. Thus, we make more experiments for this example

Table 6.12: Comparison of the objective values of Example 6.2

pop_size	P_m	P_c	generation	objective value
10	0.3	0.5	1000	15.5
10	0.2	0.3	1000	15.5
20	0.3	0.4	2000	15.5
20	0.5	0.4	2000	15.5
30	0.3	0.2	3000	15.5
30	0.4	0.2	3000	15.5

to show the robustness of the algorithm. The values of parameters and the corresponding objective values are shown in Table 6.12. It is clear that the objective values remain unchanged for different values of the parameters in the algorithm. This fact states that the hybrid intelligent algorithm is stable and robust.

Example 6.8: We also suppose that the satisfaction constraint for each route should hold at confidence level $\alpha = 0.8$. If the decision maker predetermines a time at 20, a measure-chance programming model then can be built as follows,

$$\begin{cases} \max\limits_{\mathbf{x},\mathbf{y}} \quad \min\limits_{1\le k\le 3} \mathcal{M}\{f_k(\mathbf{x},\mathbf{y},\boldsymbol{\xi}) \le 20\} \\ \text{subject to :} \\ \quad \mathcal{M}\left\{\sum\limits_{j=y_{k-1}+1}^{y_k} \eta_{x_j} \le C_k\right\} \ge 0.8, \ \text{ if } \ y_k > y_{k-1} \\ \quad k = 1,2,3 \\ \quad 1 \le x_i \le 7, \quad i = 1,2,\ldots,7 \\ \quad x_i \ne x_j, \quad i \ne j, \quad i,j = 1,2,\ldots,7 \\ \quad 0 \le y_1 \le y_2 \le 7 \\ \quad x_i, \quad i = 1,2,\ldots,7, \quad \text{integers} \\ \quad y_j, \quad j = 1,2, \quad \text{integers}. \end{cases} \tag{6.19}$$

We also set the crossover probability $P_c = 0.5$, the mutation probability $P_m = 0.3$, the population size $pop\_size = 10$, and the parameter in the rank-based evaluation function is 0.06. A run of the proposed algorithm for 1000 generations shows that the optimal solution is

$$\mathbf{x}^*=(4,2,1,5,3,7,6),$$
$$\mathbf{y}^*=(2,5).$$

In other words, the optimal operational plan is

Vehicle 1: Airport $\to 4 \to 2 \to$ Airport
Vehicle 2: Airport $\to 1 \to 5 \to 3 \to$ Airport
Vehicle 3: Airport $\to 7 \to 6 \to$ Airport

whose objective value is 0.8.

6.3 Covering Location Problem

As a valuable method for siting service facilities, facility locating is widely used in real life such as for emergency service systems. In recent decades, many researchers have studied the facility location problem (Bianchi-Church [3], Hua-Wang-Chan-Hou [36], Li-Wang-Xu [42]), and several different models have been introduced for different situations. Set covering models are the most widespread location models. The first models of this type are models that address the location set covering problem, which was introduced by Toregas-Swain-ReVelle-Bergman [62]. Another important models is to address the maximal covering location problem, which was proposed by Church-ReVelle [8].

Location Set Covering Problem

Let $N = (V, A)$, where $V = \{1, 2, \ldots, n\}$ is the set of vertices and A is the set of edges. Suppose that each vertex i of the network $N = (V, A)$ is a potential site where a facility can be located, $i = 1, 2, \ldots, n$.

The location set covering problem aims to determine the minimum number of facilities, each facility assigned to a demand vertex, such that all the demand vertices can be serviced within a maximum response time. Let t_{ij} be a response time from vertex i to j. For simplicity, we denote the network with response times as $N = (V, A; \boldsymbol{t})$, where $\boldsymbol{t} = \{t_{ij}\}$. Additionally, we ignore the response time t_{ij} by setting it to zero when $i = j$. Usually, we often set the maximum response time T, and then the set of vertices that can provide emergency service to vertex i is defined as follows,

$$S_i = \{j | t_{ji} \leq T\},$$

for $i = 1, 2, \ldots, n$.

In an uncertain environment, the response times are described as uncertain variables. Let ξ_{ij} be non-negative uncertain variables that indicate the response times from vertex i to vertex j for $i, j = 1, 2, \ldots, n$. Denote an emergency network with uncertain response times as $\mathcal{N} = (V, A; \boldsymbol{\xi})$, where $\boldsymbol{\xi} = \{\xi_{ij} | i \in V, j \in V\}$. Given a confidence level $\alpha (0 < \alpha < 1)$, Zhang-Peng-Li [77] redefined S_i as follows,

$$S_i = \{j | \mathcal{M}\{\xi_{ji} \leq T\} \geq \alpha\} \tag{6.20}$$

for $i = 1, 2, \ldots, n$.

To construct a mathematical programming model to illustrate the location set covering problem, the following decision variables are employed,

$$x_j = \begin{cases} 1, & \text{if a facility is established at vertex } j \\ 0, & \text{otherwise} \end{cases}$$

for $j = 1, 2, \ldots, n$.

Recalling that the set of potential emergency response locations of i is S_i, the service requirement can be formulated as

$$\sum_{j \in S_i} x_j \geq 1$$

for $i = 1, 2, \ldots, n$. This confirms that any demand vertex i must have at least one facility that can provide emergency service to it.

Since the location set covering problem is to minimize the total number of utilized facilities, Zhang-Peng-Li [77] formulated the location set covering problem on the network $\mathcal{N} = (V, A; \boldsymbol{\xi})$ to the following uncertain model:

$$\begin{cases} \min & \displaystyle\sum_{j=1}^{n} x_j \\ s.t. & \displaystyle\sum_{j \in S_i} x_j \geq 1, \quad i = 1, 2, \ldots, n \\ & S_i = \{j | \mathcal{M}\{\xi_{ji} \leq T\} \geq \alpha\}, \quad i = 1, 2, \ldots, n \\ & x_j \in \{0, 1\}, \quad j = 1, 2, \ldots, n. \end{cases} \tag{6.21}$$

Solution of Uncertain Location Set Covering Problem

To solve model (6.21), we must determine the elements of the set S_i for each vertex i, $i = 1, 2, \ldots, n$.

Theorem 6.4 *(Zhang-Peng-Li [77]) If ξ_{ij} are independent uncertain variables with regular uncertainty distributions Φ_{ij}, $i, j = 1, 2, \ldots, n$, then*

$$\mathcal{M}\{\xi_{ji} \leq T\} \geq \alpha$$

can be transformed as

$$T \geq \Phi_{ji}^{-1}(\alpha), \tag{6.22}$$

$i, j = 1, 2, \ldots, n$.

Proof: The theorem follows immediately from the properties of the uncertainty distribution.

Theorem 6.4 shows that S_i can be determined by

$$\{j | T \geq \Phi_{ji}^{-1}(\alpha)\}$$

for $i = 1, 2, \ldots, n$. Then model (6.21) can be transformed into the following deterministic model

$$\begin{cases} \min & \sum_{j=1}^{n} x_j \\ s.t. & \sum_{j \in S_i} x_j \geq 1, \quad i = 1, 2, \ldots, n \\ & S_i = \{j | T \geq \Phi_{ji}^{-1}(\alpha)\}, \quad i = 1, 2, \ldots, n \\ & x_j \in \{0, 1\}, \quad j = 1, 2, \ldots, n. \end{cases} \tag{6.23}$$

Additionally, Zhang-Peng-Li [77] introduced the following algorithm to search the optimal solution of model (6.21) in the uncertain network $\mathcal{N} = (V, A; \boldsymbol{\xi})$.

Algorithm 6.1 (Algorithm for model (6.21))

Step 1. Calculate $\Phi_{ij}^{-1}(\alpha)$ for each response time ξ_{ij} in $\mathcal{N} = (V, A; \boldsymbol{\xi})$.

Step 2. Determine the elements of the vertex set S_i by $\{j | T \geq \Phi_{ji}^{-1}(\alpha)\}$ for each vertex i.

Step 3. Employ some mathematical software (such as LINGO or MATLAB) to solve the 0-1 integer programming model (6.23).

Maximal Covering Location Problem

The maximal covering location problem now becomes that, given a limited number of facilities, how does one determine where they should be located so that the majority of demand is covered within a maximum response time.

As a consequence, the decision variables are given as follows

$$x_j = \begin{cases} 1, & \text{if a facility is established at vertex } j \\ 0, & \text{otherwise,} \end{cases}$$

$$y_i = \begin{cases} 1, & \text{if vertex } i \text{ is covered by a facility} \\ 0, & \text{otherwise.} \end{cases}$$

Let $\widetilde{N} = (V, A; \boldsymbol{\xi}, \boldsymbol{\eta})$ be an emergency network with uncertain response times $\boldsymbol{\xi} = \{\xi_{ij}\}$ and uncertain demands $\boldsymbol{\eta} = \{\eta_i\}$, $i, j = 1, 2, \ldots, n$. Suppose that η_i are independent uncertain variables with regular uncertainty distributions Ψ_i, $i = 1, 2, \ldots, n$. Denote $\mathrm{y} = (y_1, y_2, \ldots, y_n)$, $\mathrm{x} = (x_1, x_2, \ldots, x_n)$. The covered demand with (x, y) can be formulated as

$$\widetilde{D}(\mathrm{x}, \mathrm{y}) = \sum_{i=1}^{n} y_i \eta_i.$$

As in the uncertain location set covering problem, let S_i denote the set vertices that can provide emergency service to i. Let p be the limited number of facilities to be established. Then for the maximal covering location problem, Zhang-Peng-Li [77] shows that (x, y) is called a feasible location if and only if

$$\begin{cases} \sum\limits_{j \in S_i} x_j \geq y_i, & i = 1, 2, \ldots, n \\ \sum\limits_{j=1}^{n} x_j = p \\ S_i = \{j | \mathcal{M}\{\xi_{ji} \leq T\} \geq \alpha\}, & i = 1, 2, \ldots, n \\ x_j, y_i \in \{0, 1\}, & i, j = 1, 2, \ldots, n \end{cases}$$

where the first constraint states that the demand of vertex i is covered if there is at least one located facility with a response time that is less than the maximum response time T; The second constraint is a cardinality constraint for the total number of facilities; The third constraint is the response time constraint; Finally, the binary restriction on the decision variables are given in the fourth constraint.

Uncertainty of the Maximum Covered Demand

Note that the covered demand $\widetilde{D}(\mathrm{x}, \mathrm{y})$ in the uncertain network $\widetilde{N} = (V, A; \boldsymbol{\xi}, \boldsymbol{\eta})$ is an uncertain variable. Zhang-Peng-Li [77] discussed the uncertainty of $\widetilde{D}(\mathrm{x}, \mathrm{y})$.

It is clear that different covering level α may result in different $\widetilde{D}(\mathrm{x}, \mathrm{y})$. Let $\Upsilon_{xy|\alpha}$ denote the uncertainty distribution of $\widetilde{D}(\mathrm{x}, \mathrm{y})$ that is related to α, i.e.,

$$\Upsilon_{xy|\alpha}(d) = \mathcal{M}\{\widetilde{D}(\mathrm{x}, \mathrm{y}) \leq d\}$$

satisfying

$$S_i = \{j | \mathcal{M}\{\xi_{ji} \leq T\} \geq \alpha\}, \quad i = 1, 2, \ldots, n.$$

Since each uncertain demand η_i has a regular uncertainty distribution Ψ_i, according to Theorem 2.4, $\Upsilon_{xy|\alpha}$ is also regular. In other words, for any $\beta \in (0, 1)$,

$$\Upsilon_{xy|\alpha}^{-1}(\beta) = \sum_{i=1}^{n} y_i \Psi_i^{-1}(\beta).$$

Obviously, the maximum covered demand

$$\widetilde{D}_{\widetilde{N}} = \max_{\mathrm{x}, \mathrm{y}} \widetilde{D}(\mathrm{x}, \mathrm{y}),$$

is also an uncertain variable. Denote $\Upsilon_{\widetilde{N}|\alpha}$ as the uncertainty distribution of $\widetilde{D}_{\widetilde{N}}$ associated with

$$S_i = \{j|\mathcal{M}\{\xi_{ji} \leq T\} \geq \alpha\}, \quad i = 1, 2, \ldots, n.$$

Clearly, $\Upsilon_{\widetilde{N}|\alpha}$ is also a regular distribution. That is, for any feasible location (x, y), we have

$$\widetilde{D}_{\widetilde{N}} \geq \widetilde{D}(\mathrm{x}, \mathrm{y}),$$

and

$$\mathcal{M}\{\widetilde{D}_{\widetilde{N}} \leq d\} \leq \mathcal{M}\{\widetilde{D}(\mathrm{x}, \mathrm{y}) \leq d\}.$$

Namely, $\Upsilon_{\widetilde{N}|\alpha}(d) \leq \Upsilon_{xy|\alpha}(d)$. In addition, for any $\beta \in (0, 1)$, the following inequality holds immediately,

$$\Upsilon^{-1}_{\widetilde{N}|\alpha}(\beta) \geq \Upsilon^{-1}_{xy|\alpha}(\beta),$$

since $\Upsilon_{\widetilde{N}|\alpha}$ and $\Upsilon_{xy|\alpha}$ are both regular.

Suppose that there is a deterministic emergency network, the response time from vertex i to vertex j is denoted as t_{ij} and the amount of demand of demand vertex i is denoted as d_i. Denote D_N as the optimal covered demand in the deterministic emergency network $N = (V, A; \boldsymbol{t}, \boldsymbol{d})$, where $\boldsymbol{t} = \{t_{ij}\}, \boldsymbol{d} = \{d_i\}$. Obviously, D_N will be determined when $\boldsymbol{t}$ and $\boldsymbol{d}$ are given. In other words, D_N is a function of $\boldsymbol{t}$ and $\boldsymbol{d}$, i.e.,

$$D_N = f(\boldsymbol{t}; \boldsymbol{d}).$$

The optimal covered demand $f(\boldsymbol{t}; \boldsymbol{d})$ can then be obtained by the model as follows,

$$\begin{cases} \max & \displaystyle\sum_{i=1}^{n} y_i d_i \\ s.t. & \displaystyle\sum_{j \in S_i} x_j \geq y_i, \qquad i = 1, 2, \ldots, n \\ & \displaystyle\sum_{j=1}^{n} x_j = p \\ & S_i = \{j|t_{ji} \leq T\}, \quad i = 1, 2, \ldots, n \\ & x_j, y_i \in \{0, 1\}, \quad i, j = 1, 2, \ldots, n. \end{cases} \tag{6.24}$$

That is, $f(\boldsymbol{t}; \boldsymbol{d}) = D_N(\mathrm{x}^*, \mathrm{y}^*)$, where $(\mathrm{x}^*, \mathrm{y}^*)$ is the optimal solution of model (6.24). Clearly, $f(\boldsymbol{t}; \boldsymbol{d})$ is continuous, increasing with respect to $d_1, d_2, \ldots, d_n$. As a consequence, in the uncertain network $\widetilde{N} = (V, A; \boldsymbol{\xi}, \boldsymbol{\eta})$, given an α, we have

$$\Upsilon^{-1}_{\widetilde{N}|\alpha}(\beta) = f(\Phi^{-1}(\alpha); \Psi^{-1}(\beta))$$

for any $\beta \in (0,1)$, where $\Phi^{-1}(\alpha) = (\Phi_{ij}^{-1}(\alpha))$ and $\Psi^{-1}(\beta) = (\Psi_i^{-1}(\beta))$. It is clear that the function $f(\Phi^{-1}(\alpha); \Psi^{-1}(\beta))$ is just the optimal objective of the following model:

$$\begin{cases} \max & \displaystyle\sum_{i=1}^{n} y_i \Psi_i^{-1}(\beta) \\ s.t. & \displaystyle\sum_{j \in S_i} x_j \geq y_i, \qquad i = 1, 2, \ldots, n \\ & \displaystyle\sum_{j=1}^{n} x_j = p \\ & S_i = \{j | T \geq \Phi_{ji}^{-1}(\alpha)\}, \quad i = 1, 2, \ldots, n \\ & x_j, y_i \in \{0, 1\}, \quad i, j = 1, 2, \ldots, n. \end{cases} \tag{6.25}$$

Intuitively, Zhang-Peng-Li [77] gave the following algorithm to obtain the inverse uncertainty distribution $\Upsilon_{\widetilde{N}|\alpha}^{-1}(\beta)$ for a given confidence level α in a numerical sense,

Algorithm 6.2 (Algorithm for $\Upsilon_{\widetilde{N}|\alpha}^{-1}(\beta)$)

Step 1. Calculate the inverse uncertainty distribution $\Phi_{ji}^{-1}(\alpha)$ of ξ_{ij}. Set $t = 1$.

Step 2. Calculate the values of $\Psi_i^{-1}(0.01t)$.

Step 3. Solve model (6.25), whose optimal objective is denoted by $\Upsilon_{\widetilde{N}|\alpha}^{-1}(0.01t)$. Save $(0.01t, \Upsilon_{\widetilde{N}|\alpha}^{-1}(0.01t))$.

Step 4. If $t = 99$, stop; Otherwise, let $t \leftarrow t+1$, and then turn back to Step 2.

Uncertain Covering Location Models

To obtain the optimal plan for the maximal covering location problem in an uncertain environment, some addition criteria should be considered. In practice, the critical value criterion is usually used to optimize uncertain problems. Hence, the concept of the (α, β)-maximal covering location is defined as follows.

Definition 6.4 *(Zhang-Peng-Li [77]) Let* $(\mathrm{x}^*, \mathrm{y}^*)$ *be a feasible location; then it is called the* (α, β)*-maximal covering location if*

$$\max\left\{\bar{f} \,\middle|\, \mathcal{M}\left\{\sum_{i=1}^{n} y_i^* \eta_i \geq \bar{f}\right\} \geq \beta\right\} \geq \max\left\{\bar{f} \,\middle|\, \mathcal{M}\left\{\sum_{i=1}^{n} y_i \eta_i \geq \bar{f}\right\} \geq \beta\right\}$$

holds for any feasible location (x,y) *under the response constraint*

$$S_i = \{j|\mathcal{M}\{\xi_{ji} \leq T\} \geq \alpha\},$$

and $\alpha, \beta \in (0,1)$ *are predetermined confidence levels.*

Theorem 6.5 *(Zhang-Peng-Li [77]) Given confidence levels* α *and* β*, the* (α,β)*-maximum covering location* $(\mathrm{x}^*,\mathrm{y}^*)$ *is just the optimal plan of the deterministic emergency network* $N' = (V, A; \boldsymbol{t}', \boldsymbol{d}')$*, where* $\boldsymbol{t}' = \{t'_{ij} = \Phi_{ij}^{-1}(\alpha)|(i,j) \in A\}$ *and* $\boldsymbol{d}' = \{d'_i = \Psi_i^{-1}(1-\beta)|i \in V\}$.

Proof: According to Definition 6.4, the (α,β)-maximum covering location $(\mathrm{x}^*,\mathrm{y}^*)$ is the optimal solution of the following model:

$$\begin{cases} \max & \bar{f} \\ s.t. & \mathcal{M}\left\{\sum_{i=1}^{n} y_i\eta_i \geq \bar{f}\right\} \geq \beta \\ & \sum_{j\in S_i} x_j \geq y_i, \qquad i = 1,2,\ldots,n \\ & S_i = \{j|\mathcal{M}\{\xi_{ji} \leq T\} \geq \alpha\}, \quad i = 1,2,\ldots,n \\ & \sum_{j=1}^{n} x_j = p \\ & x_j, y_i \in \{0,1\}, \quad i,j = 1,2,\ldots,n. \end{cases} \tag{6.26}$$

It follows from the duality of the uncertain measure and the regularity of the uncertainty distributions that the first constraint

$$\mathcal{M}\left\{\sum_{i=1}^{n} y_i\eta_i \geq \bar{f}\right\} \geq \beta$$

can be equivalently transformed into

$$\mathcal{M}\left\{\sum_{i=1}^{n} y_i\eta_i \leq \bar{f}\right\} \leq 1-\beta.$$

In addition, according to Theorem 2.4, the first constraint can be reformulated as

$$\bar{f} \leq \sum_{i=1}^{n} y_i\Psi_i^{-1}(1-\beta), \tag{6.27}$$

since each η_i has a regular uncertainty distribution. Substituting expressions

(6.22) and (6.27) into model (6.26) gives a deterministic model:

$$\begin{cases} \max & \bar{f} \\ s.t. & \bar{f} \leq \sum_{i=1}^{n} y_i \Psi_i^{-1}(1-\beta) \\ & \sum_{j \in S_i} x_j \geq y_i, \qquad i = 1, 2, \ldots, n \\ & S_i = \{j | T \geq \Phi_{ji}^{-1}(\alpha)\}, \quad i = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_j = p \\ & x_j, y_i \in \{0, 1\}, \quad i, j = 1, 2, \ldots, n. \end{cases} \tag{6.28}$$

Clearly, model (6.28) is equivalent to

$$\begin{cases} \max & \sum_{i=1}^{n} y_i \Psi_i^{-1}(1-\beta) \\ s.t. & \sum_{j \in S_i} x_j \geq y_i, \qquad i = 1, 2, \ldots, n \\ & S_i = \{j | T \geq \Phi_{ji}^{-1}(\alpha)\}, \quad i = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_j = p \\ & x_j, y_i \in \{0, 1\}, \quad i, j = 1, 2, \ldots, n. \end{cases} \tag{6.29}$$

Obviously, the optimal solution to model (6.29) is just the optimal location plan of the deterministic emergency network $N' = (V, A; \boldsymbol{t}', \boldsymbol{d}')$, where $\boldsymbol{t}' = \{t'_{ij} = \Phi_{ij}^{-1}(\alpha) | (i,j) \in A\}$ and $\boldsymbol{d}' = \{d'_i = \Psi_i^{-1}(1-\beta) | i \in V\}$. Thus, the proof is completed.

A decision maker may first give a covering level $\bar{f}$ and then maximize the chance that the covered demand is greater than $\bar{f}$. If this modelling idea is approved, then the α-chance maximum covering location can be defined as follows.

Definition 6.5 *(Zhang-Peng-Li [77]) Given a covering level $\bar{f}$, the feasible location* $(\mathrm{x}^*, \mathrm{y}^*)$ *is called the α-chance maximal covering location if*

$$\mathcal{M}\left\{\sum_{i=1}^{n} y_i^* \eta_i \geq \bar{f}\right\} \geq \mathcal{M}\left\{\sum_{i=1}^{n} y_i \eta_i \geq \bar{f}\right\}$$

holds for any feasible location (x,y) *under the response constraint*

$$S_i = \{j | \mathcal{M}\{\xi_{ji} \leq T\} \geq \alpha\},$$

and $\alpha \in (0, 1)$ is a predetermined confidence level.

Theorem 6.6 *(Zhang-Peng-Li [77]) Given a confidence level α and covering level $\bar{f}$, the α-chance maximum covering location $(\mathrm{x}^*, \mathrm{y}^*)$ is just the (α, β)-maximum covering location of the uncertain emergency network $\widetilde{N} = (V, A; \boldsymbol{\xi}, \boldsymbol{\eta})$, where $\Upsilon_{\widetilde{N}|\alpha}(\bar{f}) = 1 - \beta$.*

Proof: According to Definition 6.5, the α-chance maximum covering location $(\mathrm{x}^*, \mathrm{y}^*)$ is the optimal solution to the following model:

$$\begin{cases} \max & \mathcal{M}\left\{\sum_{i=1}^{n} y_i\eta_i \geq \bar{f}\right\} \\ s.t. & \sum_{j\in S_i} x_j \geq y_i, \qquad i = 1, 2, \ldots, n \\ & S_i = \{j | \mathcal{M}\{\xi_{ji} \leq T\} \geq \alpha\}, \quad i = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_j = p \\ & x_j, y_i \in \{0, 1\}, \quad i, j = 1, 2, \ldots, n. \end{cases} \tag{6.30}$$

It is easy to verify that the above model can be reformulated to the following:

$$\begin{cases} \max & \beta \\ s.t. & \mathcal{M}\left\{\sum_{i=1}^{n} y_i\eta_i \geq \bar{f}\right\} \geq \beta \\ & \sum_{j\in S_i} x_j \geq y_i, \qquad i = 1, 2, \ldots, n \\ & S_i = \{j | \mathcal{M}\{\xi_{ji} \leq T\} \geq \alpha\}, \quad i = 1, 2, \ldots, n \\ & \sum_{j=1}^{n} x_j = p \\ & x_j, y_i \in \{0, 1\}, \quad i, j = 1, 2, \ldots, n. \end{cases} \tag{6.31}$$

As in the proof for Theorem 6.5, the first constraint of model (6.31) can be reformulated as follows

$$\bar{f} \leq \sum_{i=1}^{n} y_i \Psi_i^{-1}(1 - \beta), \tag{6.32}$$

since each η_i has a regular uncertainty distribution. Based on expressions

(6.22) and (6.32), model (6.31) is equivalent to the following model

$$\begin{cases} \max \quad \beta \\ s.t. \quad \bar{f} \leq \sum_{i=1}^{n} y_i \Psi_i^{-1}(1-\beta) \\ \qquad \sum_{j \in S_i} x_j \geq y_i, \qquad i = 1, 2, \ldots, n \\ \qquad S_i = \{j | T \geq \Phi_{ji}^{-1}(\alpha)\}, \quad i = 1, 2, \ldots, n \\ \qquad \sum_{j=1}^{n} x_j = p \\ \qquad x_j, y_i \in \{0, 1\}, \quad i, j = 1, 2, \ldots, n. \end{cases} \tag{6.33}$$

Let $1 - \overline{\beta} = \Upsilon_{\widetilde{N}|\alpha}(\bar{f})$. Then, $\Upsilon_{\widetilde{N}|\alpha}^{-1}(1-\overline{\beta}) = \bar{f}$ since $\Upsilon_{\widetilde{N}|\alpha}$ is a regular uncertainty distribution. For the given confidence levels α and $\overline{\beta}$, the $(\alpha, \overline{\beta})$-maximal covering location $(\mathrm{x}^*, \mathrm{y}^*)$ can be obtained based on Theorem 6.5. Obviously, $(\mathrm{x}^*, \mathrm{y}^*)$ and $\overline{\beta}$ satisfy the following constraints

$$\begin{cases} \bar{f} \leq \sum_{i=1}^{n} y_i \Psi_i^{-1}(1-\overline{\beta}) \\ \sum_{j \in S_i} x_j \geq y_i, \qquad i = 1, 2, \ldots, n \\ S_i = \{j | T \geq \Phi_{ji}^{-1}(\alpha)\}, \quad i = 1, 2, \ldots, n \\ \sum_{j=1}^{n} x_j = p \\ x_j, y_i \in \{0, 1\}, \quad i, j = 1, 2, \ldots, n. \end{cases}$$

That is, $(\mathrm{x}^*, \mathrm{y}^*)$ is a feasible location to model (6.33), and $\overline{\beta}$ is the corresponding value of the objective function. In fact, $\overline{\beta}$ is also the optimal objective value.

For any $\beta > \overline{\beta}$, $\beta \in (0, 1)$, we have

$$\Upsilon_{\widetilde{N}|\alpha}^{-1}(1-\beta) < \Upsilon_{\widetilde{N}|\alpha}^{-1}(1-\overline{\beta}) = \bar{f}.$$

More precisely,

$$\sum_{i=1}^{n} y_i \Psi_i^{-1}(1-\beta) < \bar{f}. \tag{6.34}$$

which leads to a contradiction with the constraints in model (6.33).

In other words, $\overline{\beta}$ is the optimal value of the objective function, and $(\mathrm{x}^*, \mathrm{y}^*)$ is the optimal solution of model (6.33). Thus, the proof is completed.

Table 6.13: The response time ξ_{ij}

i \ j	1	2	3	4	5	6	7
1	0	$\mathcal{L}(2,6)$	$\mathcal{L}(4,7)$	$\mathcal{L}(3,6)$	$\mathcal{L}(2,4)$	$\mathcal{L}(3,5)$	$\mathcal{L}(3,6)$
2	$\mathcal{L}(2,6)$	0	$\mathcal{L}(3,5)$	$\mathcal{L}(2,6)$	$\mathcal{L}(3,7)$	$\mathcal{L}(4,6)$	$\mathcal{L}(3,5)$
3	$\mathcal{L}(3,7)$	$\mathcal{L}(3,5)$	0	$\mathcal{L}(4,8)$	$\mathcal{L}(6,8)$	$\mathcal{L}(5,7)$	$\mathcal{L}(4,6)$
4	$\mathcal{L}(3,5)$	$\mathcal{L}(2,4)$	$\mathcal{L}(3,7)$	0	$\mathcal{L}(2,6)$	$\mathcal{L}(1,5)$	$\mathcal{L}(2,6)$
5	$\mathcal{L}(2,4)$	$\mathcal{L}(4,6)$	$\mathcal{L}(3,9)$	$\mathcal{L}(1,3)$	0	$\mathcal{L}(2,6)$	$\mathcal{L}(3,5)$
6	$\mathcal{L}(5,7)$	$\mathcal{L}(2,6)$	$\mathcal{L}(3,5)$	$\mathcal{L}(3,6)$	$\mathcal{L}(2,7)$	0	$\mathcal{L}(3,7)$
7	$\mathcal{L}(2,4)$	$\mathcal{L}(4,8)$	$\mathcal{L}(4,6)$	$\mathcal{L}(2,7)$	$\mathcal{L}(5,7)$	$\mathcal{L}(3,6)$	0

Table 6.14: The demand η_i for vertex i

vertex i	demand η_i
1	$\mathcal{L}(12,16)$
2	$\mathcal{L}(15,20)$
3	$\mathcal{L}(13,17)$
4	$\mathcal{L}(13,15)$
5	$\mathcal{L}(12,18)$
6	$\mathcal{L}(14,18)$
7	$\mathcal{L}(12,14)$

Numerical Examples

Suppose that there are 7 emergency demand vertices in an emergency network. Our goal is to determine the best locations for some facilities. In order to make an optimal decision, the response times and the demands are evaluated based on some experts' experience, and are assumed to be independent linear uncertain variables, which are listed in Table 6.13 and Table 6.14, respectively.

Example 6.9: Let $\alpha = 0.9$, $T = 5$. If we want to determine the minimum number of facilities such that all the demand vertices can be met within a maximum response time, then an uncertain location set covering model can be proposed as model (6.21).

Algorithm 6.1 shows that the optimal solution vector x^* is $(0, 0, 0, 1, 1, 1, 0)$. That is, we should locate the facilities at demand vertices 4, 5 and 6; then, any demand vertex can be serviced by at least one facility.

As we know, different α may result in different location plans and different objective values. Thus, we take more numerical experiments with different α, and the results are listed in Table 6.15. Clearly, the optimal objective value is non-decreasing with respect to α.

Example 6.10: Let $\alpha = 0.9$, $\beta = 0.95$, $T = 5$ and $p = 2$. If we want to cover

Table 6.15: Comparison of the optimal objective values of Example 6.3 with different α

α	Location plan x^*	Objective value
0.1	(0,1,0,0,0,0,0)	1
0.2	(0,1,0,0,0,0,0)	1
0.3	(0,1,0,0,0,0,0)	1
0.4	(0,0,0,1,0,0,0)	1
0.5	(0,0,0,1,0,0,0)	1
0.6	(0,1,0,0,0,1,0)	2
0.7	(0,0,0,0,1,1,0)	2
0.8	(0,0,0,1,1,1,0)	3
0.9	(0,0,0,1,1,1,0)	3

Table 6.16: Comparison of the optimal objective values of Example 6.3 with different β

$(\alpha, 1-\beta)$	Objective value
(0.9, 0.1)	81.1
(0.9, 0.2)	83.2
(0.9, 0.3)	85.5
(0.9, 0.4)	88
(0.9, 0.5)	90.5
(0.9, 0.6)	93
(0.9, 0.7)	95.5
(0.9, 0.8)	98
(0.9, 0.9)	100.5

the greatest demands for a limited number of facilities, then an uncertain maximum covering location model can be proposed as model (6.26).

According to Theorem 6.4, the optimal solution can be obtained. That is, $x^* = (0, 1, 0, 1, 0, 0, 0)$, $y^* = (1, 1, 1, 1, 0, 1, 1)$. This means that if we locate the facilities at demand vertices 2 and 4, then the demand vertices 1, 2, 3, 4, 6, and 7 can be covered, whose objective value is 80.05. Although it cannot cover all the demand vertices, the covered demand will be the maximum within the limited number of facilities.

Given $\alpha = 0.9$, we obtain Table 6.16 for different β. It follows from Algorithm 6.3 that the distribution function $\Upsilon_{\widetilde{N}|0.9}$ can be obtained, which is drawn by using MATLAB in Figure 6.1.

Example 6.11: Let $\alpha = 0.9$, $T = 5$ and $p = 2$. Given a covering level $\overline{f} = 85.5$, if we want to maximize the chance of the covered demand being greater than $\overline{f}$ for a limited number of facilities, then an uncertain maximum covering location model can be proposed as model (6.30).

To obtain the 0.9-chance maximal covering location, we should first calcu-

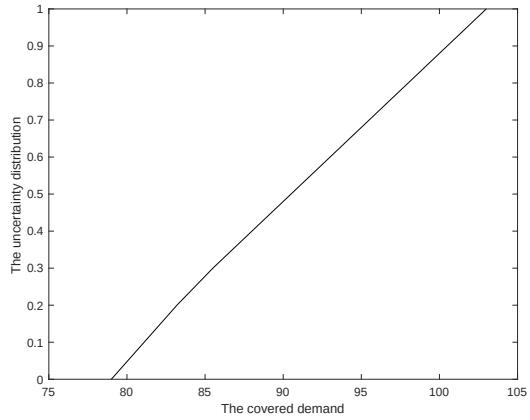


Figure 6.1: Distribution function $\Upsilon_{\widetilde{N}|0.9}$ of $\widetilde{D}_{\widetilde{N}}$ that is related to $\alpha = 0.9$

late $1 - \beta = \Upsilon_{\widetilde{N}|0.9}(85.5)$. According to Table 6.16, we know $\Upsilon_{\widetilde{N}|0.9}(85.5) = 0.3$. In other words, $\beta = 0.7$. It follows from Theorem 6.6 that the 0.9-chance maximal covering location that we want to obtain is just the $(0.9, 0.7)$-maximal covering location. Thus, the optimal location plan of Example 6.3, i.e., the 0.9-chance maximal covering location is $\mathrm{x}^* = (1, 1, 0, 0, 0, 0, 0)$, $\mathrm{y}^* = (1, 1, 1, 0, 1, 1, 1)$.

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List of Frequently Used Symbols

$\mathcal{M}$	uncertain measure
$(\Gamma, \mathcal{L}, \mathcal{M})$	uncertainty space
ξ, η, τ	uncertain variables
Φ, Ψ, Υ	uncertainty distributions
Φ^{-1}, Ψ^{-1}, Υ^{-1}	inverse uncertainty distributions
$\mathcal{L}(a, b)$	linear uncertain variable
$\mathcal{Z}(a, b, c)$	zigzag uncertain variable
$\mathcal{N}(e, \sigma)$	normal uncertain variable
$\mathcal{LOGN}(e, \sigma)$	lognormal uncertain variable
E	expected value
V	variance
$\Re$	the set of real numbers
$\vee$	maximum operator
$\wedge$	minimum operator

Index

Bo Zhang and Jin Peng
Uncertain Graphs and Network Optimization

In real life world, there usually exist indeterminacy relationships between our research objects. Uncertain graph provides a theoretical tool to describe this type of uncertainty background.

Uncertain network optimization is the study of network optimization with uncertainty which addresses new optimization methodology to a various of network optimization problems in uncertain environment.

This is an introductory textbook on uncertain graphs, uncertain network, and uncertain network optimization. This textbook also shows applications of uncertain network optimization to a lot of real problems such as transportation problem, dispatching medical supplies problem, and location selection problem.

The book is suitable for researchers, engineers, and students in the field of mathematical science, information science, computer science, decision science, management science and engineering, control science and engineering, artificial intelligence, economics, industrial engineering, and operations research.